Generalized Derivatives for Nonsmooth Optimization

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Why *Nonsmooth* Optimization?

- Many optimization problems involve nondifferentiable terms:

$$\min_x |x|, \quad \min_x \|x\|_1, \quad \min_x f(x) + \iota_C(x)$$

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Goal: Extend derivative concepts to nonsmooth settings.

Outline

First-order Concepts

Convex Subdifferential(s)

Clarke Generalization

Subgradient-based Algorithm

Second-order Concepts

General Subdifferential

Second-order Subderivative and Optimality

Newton Derivative

Semismooth Newton Method

Summary

Basic Notions

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or *locally Lipschitz* (at a point $x_0 \in \mathbb{R}^n$):

$$\text{there is } \mathcal{U} \ni x_0,\ L > 0 \text{ s.t. } |f(x) - f(y)| \leq L\|x - y\| \text{ for all } x, y \in \mathcal{U}$$

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Convex Subdifferential

For a proper convex $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $x \in \operatorname{dom} f$

$$\partial f(x) := \{s \in \mathbb{R}^n \mid f(y) \geq f(x) + (s\,,\, y - x) \quad \forall y \in \mathbb{R}^n\}\,,$$

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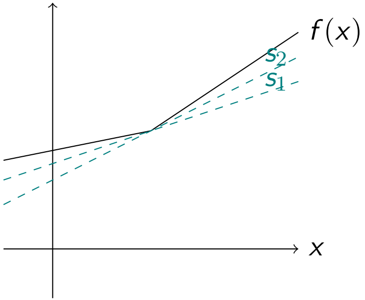

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- $0 \in \partial f(x^*) \iff x^*$ is a minimizer.
- $\partial f(x)$ always nonempty if f is convex and $x \in \operatorname{int} \operatorname{dom} f$ or f is l.s.c.

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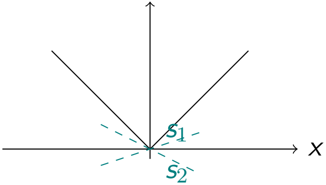

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Also known as the (convex) *normal cone* $\mathcal{N}_{\mathcal{C}}(x)$ of $\mathcal{C}$ at $x \in \mathbb{R}^n$.

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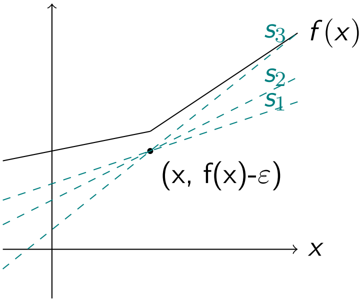

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- can be easily approximated by polyhedral sets

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$$\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x),$$

  with equality if, e. g., there is a $x_0 \in \operatorname{int} \operatorname{dom} f \cap \operatorname{int} \operatorname{dom} g$

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- Additionally: $\partial(f+g)(x) = \bigcap_{\varepsilon>0} \overline{\partial_\varepsilon f(x) + \partial_\varepsilon g(x)}$,
- Chain rule: for a convex, increasing map $h\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}^n$

  $$\partial(h \circ f) = \partial h(f(x))\partial f(x) = \{sv \mid s \in \partial h(f(x)),\ v \in \partial f(x)\}$$

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- If x^* is a *local* minimizer of f, then $0 \in \partial^\circ f(x^*)$.

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- Lebourg's mean value theorem: for any $x, y \in \mathbb{R}^n$ there exist $z \in (x, y)$ and $v \in \partial^\circ f(z)$ such that

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- Convex hull characterization:

$$\partial^\circ f(x) = \mathrm{conv}\{v \in \mathbb{R}^n \mid v = \lim_{k\to\infty} \nabla f(x^{(k)}),\ x^{(k)} \to x,\ x^{(k)} \in S\}$$

where S is the set of points where f is differentiable.

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while *convergence criterion is not fulfilled* **do**

  Compute $g^{(k)} \in \partial f(x^{(k)})$

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  Set $k := k + 1$

end

Result: $x^{(k_*)}$

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- Sublinear convergence to (approximate) minimizer if f is convex
- Convergence to stationary points if f is locally Lipschitz

Second-Order Motivation

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  - Necessary optimality conditions, only sufficient for convexity.
- **Second-order generalized derivatives** provide curvature information.
  - This enables second-order necessary and sufficient optimality conditions for nonconvex functions.

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$$f(y) \geq f(x) + (v\,,\, y - x) + o(\|y - x\|),$$

i. e.

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$$f(y) \geq f(x) + (v\,,\, y - x) + o(\|y - x\|),$$

i. e.

$$\liminf_{y \to x, y \neq x} \frac{f(y) - f(x) - (v\,,\, y - x)}{\|y - x\|} \geq 0,$$

or a *general subgradient* of f at x, $v \in \partial f(x)$, if there are sequences $x^{(k)} \to_f x$ and $v^{(k)} \in \hat{\partial} f(x^{(k)})$ such that $v^{(k)} \to v$.

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General Subdifferential: Example

Consider the function

$$f(x) = \begin{cases} x^2 + x & \text{if } x \leq 0, \\ 1 - x & \text{if } x > 0. \end{cases}$$

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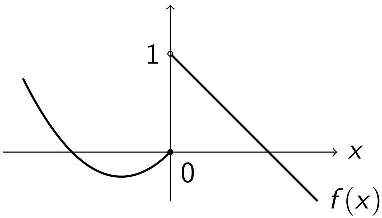

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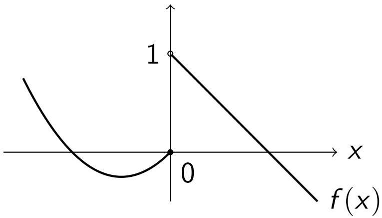

We use the definition with $x^{(k)} = -1/k \to 0$ and find

$$\hat{\partial} f(0) = \partial f(0) = [1, +\infty).$$

General Subdifferential: Example II

Let $f : \mathbb{R} \to \mathbb{R}$ be the function

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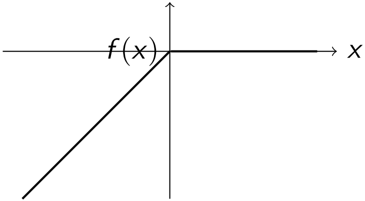

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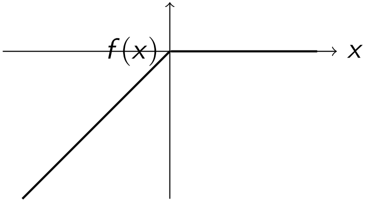

At $x = 0$, using the definition $\hat{\partial} f(0) = \emptyset$, $\partial f(0) = \{0, 1\}$, so that $\partial^\circ f(0) = [0, 1]$.

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Necessary and sufficient optimality conditions

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- If $0 \in \partial f(x^*)$ and $d^2 f(x^*|0)(w) > 0$ for all $w \neq 0$, then x^* is a local minimizer of f.

Newton Derivative

A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is *Newton differentiable* at $x \in \mathbb{R}^m$ with Newton derivative $D_N f(x)$ if

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- The chain rule applies
- If f is piecewise differentiable, then it is Newton differentiable everywhere and every $D_N f(x) \in \partial^\circ f(x)$ is a Newton derivative.

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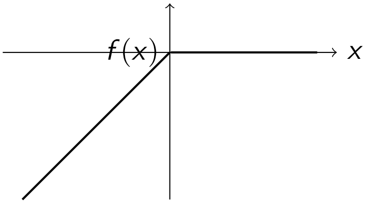

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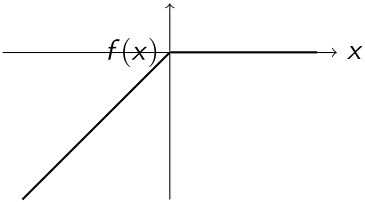

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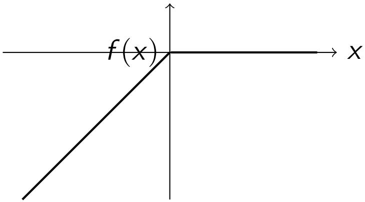

At $x = 0$, we've seen $\hat{\partial} f(0) = \emptyset$, $\partial f(0) = \{0, 1\}$, and $\partial^\circ f(0) = [0, 1]$. Then its Newton derivative is, for $\delta \in [0, 1]$

$$D_N f(x)h = \begin{cases} h & \text{if } x < 0, \\ \delta h & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases} .$$

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Goal: Find a zero of a function $f : \mathbb{R}^m \to \mathbb{R}^n$

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```
while convergence criterion is not fulfilled do
    Solve D_N f(x^(k)) g^(k) = -f(x^(k)) for g^(k)
    Set x^(k+1) = x^(k) + g^(k)
    Set k := k + 1
end
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Result: $x^{(k_*)}$

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- Local superlinear convergence, i. e. $\lim_{k \to +\infty} \frac{x^{(k+1)} - x^*}{x^{(k)} - x^*} = 0$, to a x^* such that $f(x^*) = 0$.

Summary

- **Convex subdifferential**
- **Generalized gradient for locally Lipschitz functions**
- **Subgradient-based algorithm**
- **Second-order generalized derivatives and optimality**
- **Semismooth Newton algorithm**

See you later for more on manifolds!