NTNU

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The Riemannian Convex Bundle Method

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joint work with

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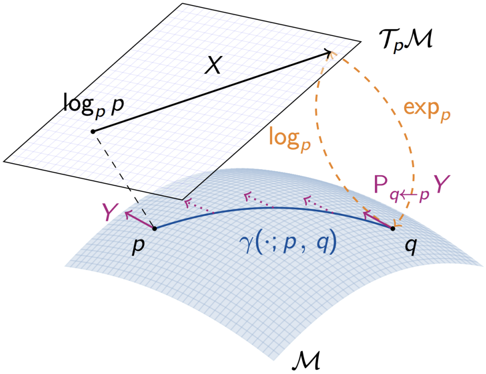
Riemannian Geometry

Notation

- Smooth Riemannian manifold $\mathcal{M}$
- Tangent space $\mathcal{T}_p\mathcal{M}$ at the point $p \in \mathcal{M}$
- Inner product $(\cdot,\cdot)_p : \mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \to \mathbb{R}$
- Exponential map $\exp_p X_p = \gamma_{pq}(1) = q$
- Logarithmic map $\log_p q = \exp_p^{-1} q = X_p$
- Parallel transport $\mathrm{P}_{q\leftarrow p} : \mathcal{T}_p\mathcal{M} \to \mathcal{T}_q\mathcal{M}$



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- $f\colon \mathcal{M} \to \overline{\mathbb{R}}$ is such that
  - $\operatorname{dom} f$ is a nonempty, bounded, strongly geodesically convex set with nonempty interior in $\mathcal{M}$
  - geodesically convex on $\operatorname{dom} f$: $f \circ \gamma\colon [0, 1] \to \overline{\mathbb{R}}$ for all geodesic arcs $\gamma \subseteq \operatorname{dom} f$ is convex in the usual sense
  - lower semi-continuous: $\liminf_{q \to p} f(q) \geq f(p)$

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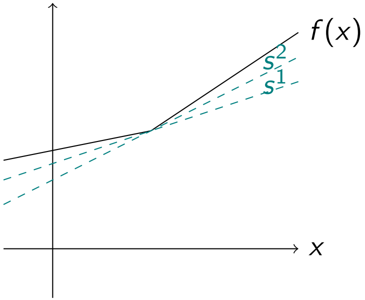
Goal. Solve this non-smooth optimization problem with a convex bundle method.

The Convex Subdifferential(s)

For a convex function, the subdifferential is defined as

$$\partial f(x) = \left\{ s \in \mathbb{R}^n \;\middle|\; f(y) \geq f(x) + (s)^T (y - x) \text{ for all } y \in \mathbb{R}^n \right\}$$

and it is a non-empty, closed and convex subset.



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$$\partial_\varepsilon f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + (s)^T (y - x) - \varepsilon \text{ for all } y \in \mathbb{R}^n \right\}$$

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- keeps memory of the last “best” points
- solves a stabilization subproblem
  - we employ a dual approach with the $\varepsilon-$subdifferential as in Geiger and Kanzow 2002
- generates sequences of *candidate* points and *stability centers*

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$$G_\varepsilon^k := \left\{ \sum_{j=1}^{k} \lambda_j s^j \;\middle|\; \sum_{j=1}^{k} \lambda_j \, e_j^k \leq \varepsilon, \; \sum_{j=1}^{k} \lambda_j = 1, \; \lambda_j \geq 0 \text{ for all } j = 1, \dots, k \right\}$$

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- Main challenge on manifolds: given $p^1, \dots, p^k \in \mathcal{M}$, and $X_{p^j} \in \partial f(p^j)$, then

$$\mathrm{P}_{p^k \leftarrow p^j} X_{p^j} \in \partial_c f(p^k) \quad \text{for some} \quad c > 0?$$

Curvature Correction

Let $\delta := \operatorname{diam}(\operatorname{dom} f) < +\infty$. Define

$$\zeta_{1,\omega}(\delta) := \begin{cases} 1 & \text{if } \omega \geq 0, \\ \sqrt{-\omega}\,\delta \coth(\sqrt{-\omega}\,\delta) & \text{if } \omega < 0, \end{cases}$$

$$\zeta_{2,\Omega}(\delta) := \begin{cases} 1 & \text{if } \Omega \leq 0, \\ \sqrt{\Omega}\,\delta \cot(\sqrt{\Omega}\,\delta) & \text{if } \Omega > 0, \end{cases}$$

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$$c_j^k := f(p^k) - f(p^j) - \Bigl(X_{p^j}, \log_{p^j} p^k\Bigr) + \varrho \|\log_{p^j} p^k\| \|X_{p^j}\|,$$

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with $X_{p_j} \in \partial f(p^j)$, we get

$$G_\varepsilon^k := \left\{ \sum_{j=1}^k \lambda_j \mathrm{P}_{p^k \leftarrow p^j} X_{p^j} \;\middle|\; \sum_{j=1}^k \lambda_j\, c_j^k \leq \varepsilon,\ \sum_{j=1}^k \lambda_j = 1,\ \lambda_j \geq 0 \text{ for all } j = 1, \ldots, k \right\}$$

with $G_\varepsilon^k \subseteq \partial_\varepsilon f(p^k)$, and $\mathrm{P}_{p^k \leftarrow p^j} X_{p^j} \in \partial_{c_j^k} f(p^k)$.

The Euclidean Subproblem

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Data: $x^1 = y^1 \in \mathbb{R}^n$, $g^0 = s^1 \in \partial f(x^1)$, $m \in (0,1)$, $\varepsilon_0 = e_1^1 = 0$, $k = 1$, $J_k = \{1\}$

while *not converged* **do**

Compute a solution $\lambda^k \in \mathbb{R}^{|J_k|}$ of the subproblem and set

$$g^k := \sum_{j \in J_k} \lambda_j^k s^j, \quad \varepsilon_k := \sum_{j \in J_k} \lambda_j^k e_j^k, \quad d^k := -g^k, \quad \xi_k := -\|g^k\|^2 - \varepsilon_k$$

Set $y^{k+1} := x^k + d^k$

If $f(y^{k+1}) \leq f(x^k) + m\xi_k$, then set $x^{k+1} := y^{k+1}$, else set $x^{k+1} := x^k$

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Result: x^{k_*}, for some $k_* \in \mathbb{N}$.

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Set $t_k := 1$. While $q^{k+1} := \exp_{p^k}(t_k d^k) \notin \operatorname{int}(\operatorname{dom} f)$ and $q^{k+1} \neq p^k$
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Result: x^{k_*}, for some $k_* \in \mathbb{N}$.

The Riemannian Convex Bundle Method

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Convergence

- In the Euclidean case we have Geiger and Kanzow 2002, Theorem 6.80.

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Let the solution set $S = \{x^* \in \mathbb{R}^n \mid f(x^*) = \inf_{x \in \mathbb{R}^n} f(x)\}$ *of the minimization problem be nonempty. Then every sequence* $\{x^k\}$ *generated by the bundle method algorithm converges to a minimizer of* f.

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- In the Riemannian case, we have obtained convergence in case the number of serious steps generated by the algorithm is infinite.
- The case where only a finite number of serious steps is generated is still *open*. The main problem is studying the convergence of the curvature-dependent term $\varrho \|\log_{q^j} p^{k+1}\| \|X_{q^j}\|$.

Convergence

In Geiger and Kanzow 2002, Theorem 6.80, they are able to show that

$$\sum_{j=k_*}^{+\infty} \left(\|g^j\|^2 + \varepsilon_j\right)^2 < +\infty$$

where $k_* \in \mathbb{N}$ is the index that corresponds to the last serious iterate, namely

$$p^k = p^{k_*} \quad \text{for all } k \geq k_*.$$

This is possible because from the definition of e_j^k one gets a bound

$$\left(s^k\,,\, g^{k-1}\right) < m\left(\|g^{k-1}\|^2 + \varepsilon_{k-1}\right) - e_k^k,$$

whereas in the Riemannian case one has

$$\left(\mathrm{P}_{p^k \leftarrow q^k} X_{q^k}\,,\, t_{k-1} g^{k-1}\right) < m\left(\|g^{k-1}\|^2 + \varepsilon_{k-1}\right) - c_k^k + \varrho \,\|t_{k-1} g^{k-1}\|\,\|X_{q^k}\|.$$

Implementation

The algorithm is implemented[1] in Julia using `Manopt.jl` (Bergmann 2022) which uses manifolds from `Manifolds.jl` (Axen et al. 2021). A solver call just looks like

```
p* = convex_bundle_method(M, f, ∂f, p0)
```

where

- `M` is a Riemannian manifold
- `f` is the objective function
- `∂f` is a subgradient of the objective function
- `p0` is an initial point on the manifold
- the parameter for the descent test is set at a default $m = 0.0125$

The default stopping criterion for the algorithm is set to

$$-\xi_k \leq 10^{-8}.$$

[1] not yet in a release version

Numerical Example: Signal Denoising on $\mathcal{H}^2$

Let $\mathcal{H}^2$ be the two-dimensional hyperbolic space and let $n = 496$. By projecting a square wave onto $\mathcal{H}^2$, we manufacture an artificial signal which can be interpreted as a point $q \in (\mathcal{H}^2)^n$.

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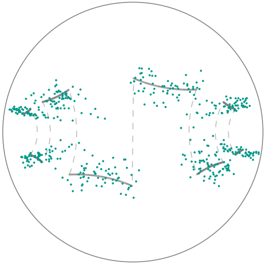

Figure: Artificial signal $q \in (\mathcal{H}^2)^{496}$ in gray, and noisy data $\bar{q} \in (\mathcal{H}^2)^{496}$ in teal.

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$$f_q(p) = \frac{1}{n}\big(g(p, q) + \alpha\,\mathrm{TV}(p)\big)$$

where

$$g(p, q) = \frac{1}{2}\sum_{i=1}^{n} \mathrm{dist}(p^i, q^i)^2 \quad \text{and} \quad \mathrm{TV}(p) = \sum_{i=1}^{n-1} \mathrm{dist}(p^i, p^{i+1}).$$

We also set $\mathrm{diam}(\mathrm{dom}\, f) := \frac{5}{2}\,\mathrm{dist}(q, \bar{q})$.

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Finally, we compare the Riemannian Convex Bundle Method (RCBM) to the Proximal Bundle Algorithm (PBA), the Subgradient Method (SGM), and the Cyclic Proximal Point Algorithm (CPPA).

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Algorithm	Iter.	Time (sec.)	Objective	Error
RCBM	3232	94.290	$1.7929 \cdot 10^{-3}$	$3.3101 \cdot 10^{-4}$
PBA	14 879	79.036	$1.8160 \cdot 10^{-3}$	$4.2182 \cdot 10^{-4}$
SGM	15 000	78.742	$1.7918 \cdot 10^{-3}$	$3.3004 \cdot 10^{-4}$
CPPA	15 000	73.065	$1.7928 \cdot 10^{-3}$	$3.3229 \cdot 10^{-4}$

Comparisons between the four algorithms on $(\mathcal{H}^2)^{496}$ for a TV parameter of $\alpha = 0.05$.

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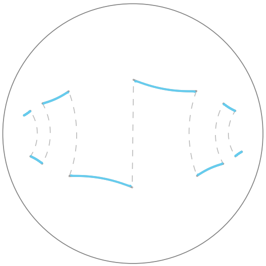

Figure: Denoised reconstruction $q^* \in (\mathcal{H}^2)^{496}$ in cyan.

Numerical Example: Riemanniann Median on $\mathcal{S}^d$

Let $\mathcal{S}^d$ be the $d-$dimensional sphere and let $q^1, \ldots, q^n \in \mathcal{S}^d$ be $n = 1000$ Gaussian random data points sampled in a ball of radius $\frac{\pi}{3}$ around the north pole $\bar{p}$, $B_{\frac{\pi}{3}}(\bar{p})$.

NTNU

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$$f(p) = \begin{cases} \frac{1}{n} \sum_{j=1}^{n} \operatorname{dist}(p, q^j) & \text{if } p \in B_{\frac{\pi}{3}}(\bar{p}), \\ +\infty & \text{otherwise.} \end{cases}$$

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Consider now the dataset

$$\mathcal{D} = \left\{ q^1, \ldots, q^n \,\middle|\, q^j \in \mathcal{S}^d \text{ for all } j = 1, \ldots, n \right\}.$$

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$$\mathcal{D} = \left\{ q^1, \dots, q^n \,\middle|\, q^j \in \mathcal{S}^d \text{ for all } j = 1, \dots, n \right\}.$$

The Riemannian geometric median p^* of $\mathcal{D}$ is then defined as

$$p^* := \operatorname*{arg\,min}_{p \in \mathcal{S}^d} f(p).$$

The initial point is chosen as one of the two points that realize the maximal distance within $\mathcal{D}$.

Numerical Example: Riemanniann Median on $\mathcal{S}^d$

	RCBM			PBA		
Dimension	Iter.	Time (sec.)	Objective	Iter.	Time (sec.)	Objective
2	16	$4.74 \cdot 10^{-3}$	0.393 47	31	$7.80 \cdot 10^{-3}$	0.393 47
4	77	$2.85 \cdot 10^{-2}$	0.391 46	71	$2.29 \cdot 10^{-2}$	0.391 46
32	20	$1.32 \cdot 10^{-2}$	0.393 48	19	$1.09 \cdot 10^{-2}$	0.393 48
1024	30	$3.06 \cdot 10^{-1}$	0.402 59	42	$4.16 \cdot 10^{-1}$	0.402 59
32 768	50	$1.85 \cdot 10^{1}$	0.391 92	78	$2.64 \cdot 10^{1}$	0.391 92

	SGM		
Dimension	Iter.	Time (sec.)	Objective
2	4570	$2.87 \cdot 10^{-2}$	0.393 47
4	5000	1.33	0.391 46
32	5000	2.16	0.393 48
1024	4646	$4.31 \cdot 10^{1}$	0.402 59
32 768	75	$2.26 \cdot 10^{1}$	0.391 92

Comparisons between the three algorithms on S^d with varying dimension.

Conclusion and Future Work

In summary:

- presented the bundle method for geodesically convex functions on Riemannian manifolds
- touched upon convergence and the problems therein
- showed two numerical examples

To do:

- further investigate convergence

Bibliography

- Alimisis, Foivos et al. (2021). "Momentum improves optimization on Riemannian manifolds". In: *Proceedings of the 24th International Conference on Artificial Intelligence and Statistics*. Ed. by Arindam Banerjee and Kenji Fukumizu. Vol. 130. Proceedings of Machine Learning Research. PMLR, pp. 1351–1359. URL: https://proceedings.mlr.press/v130/alimisis21a.html.
- Axen, Seth D. et al. (2021). *Manifolds.jl: An Extensible Julia Framework for Data Analysis on Manifolds*. arXiv: 2106.08777.
- Bergmann, Ronny (2022). "Manopt.jl: Optimization on Manifolds in Julia". In: *Journal of Open Source Software* 7.70, p. 3866. DOI: 10.21105/joss.03866.
- Bonnans, J.-F. et al. (2006). *Numerical optimization: theoretical and practical aspects*. Springer-Verlag.
- Geiger, C. and C. Kanzow (2002). *Theorie und Numerik restringierter Optimierungsaufgaben*. New York: Springer. DOI: 10.1007/978-3-642-56004-0.