NTNU

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# The Convex Bundle Method on Hadamard Manifolds

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joint work with

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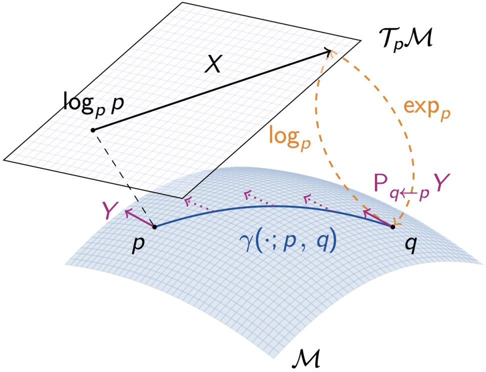
# Riemannian Geometry

Notation

- Smooth Riemannian manifold $\mathcal{M}$
- Tangent space $\mathcal{T}_p\mathcal{M}$ at the point $p \in \mathcal{M}$
- Inner product $\langle\cdot,\cdot\rangle_p\colon \mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \to \mathbb{R}$
- Exponential map $\exp_p X_p = \gamma_{pq}(1) = q$
- Logarithmic map $\log_p q = \exp_p^{-1} q = X_p$
- Parallel transport $\mathrm{P}_{q\leftarrow p}\colon \mathcal{T}_p\mathcal{M} \to \mathcal{T}_q\mathcal{M}$



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  - lower semi-continuous: $\liminf_{q \to p} f(q) \geq f(p)$

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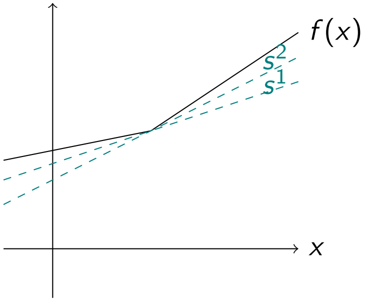
**Goal**. Solve this non-smooth optimization problem with the bundle method

# The Subdifferentials

For a convex function, the subdifferential is defined as

$$\partial f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + (s)^T(y - x) \text{ for all } y \in \mathbb{R}^n \right\}$$

and it is a non-empty, closed and convex subset.



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$$\partial_\varepsilon f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + (s)^T (y - x) - \varepsilon \text{ for all } y \in \mathbb{R}^n \right\}$$

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- keeps memory of the last “best” points
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  - we employ a dual approach with the $\varepsilon-$subdifferential as in Geiger and Kanzow 2002
- generates sequences of *candidate* points and *stability centers*

# Approximating the $\varepsilon-$subdifferential

Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Given $x_1, \ldots, x_k \in \mathbb{R}^n$, and $s^j \in \partial f(x_j)$, then

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for all $p_{j+2} \in B_\delta(p_j) \cup B_\delta(p_{j+1})$ and all $X_{p_j} \in \mathcal{T}_{p_j}\mathcal{M}$ (Azagra and Ferrera 2005, p. 169). We localize by choosing a set $\mathcal{U}$ with $\varrho = \sqrt{2}\,\mathrm{diam}(\mathcal{U}) < +\infty$.

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- Enlarge the $\varepsilon-$subdifferential by adding $\varrho\sqrt{\sigma}\|X_{p_j}\|$:
$$c_j^k = f(p_k) - f(p_j) - \langle X_{p_j}, \log_{p_j} p_k \rangle + \varrho\sqrt{\sigma}\|X_{p_j}\|$$

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**Data:** $x_1 = y_1 \in \mathbb{R}^n$, $g^0 = s^1 \in \partial f(x_1)$, $m \in (0,1)$, $\varepsilon^0 = e_1^1 = 0$, $k = 1$, $J_k = \{1\}$

Compute a solution $\lambda^k \in \mathbb{R}^{|J_k|}$ of the stabilizing subproblem and set

$$g^k = \sum_{j \in J_k} \lambda_j^k s^j, \quad \varepsilon^k = \sum_{j \in J_k} \lambda_j^k e_j^k, \quad d^k = -g^k, \quad \xi^k = -\|g^k\|^2 - \varepsilon^k$$

**while** *not converged* **do**

Set $y_{k+1} = x_k + d^k$ and compute $s^{k+1} \in \partial f(y_{k+1})$

If $f(y_{k+1}) \leq f(x_k) + m\xi^k$, then set $x_{k+1} = y_{k+1}$, else set $x_{k+1} = x_k$

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**Result:** $\{x_k\}_{k \in \mathbb{N}}$, $\{y_k\}_{k \in \mathbb{N}}$

# The Convex Hadamard Bundle Method

**Data:** $p_1 = q_1 \in \mathcal{M}$, $g^0 = X_1 \in \partial f(p_1)$, $m \in (0,1)$, $\varepsilon^0 = c_1^1 = 0$, $k = 1$, $J_k = \{1\}$

Compute a solution $\lambda^k \in \mathbb{R}^{|J_k|}$ of the stabilizing subproblem and set

$$g^k = \sum_{j \in J_k} \lambda_j^k \mathrm{P}_{p_k \leftarrow q_j} X_{q_j}, \quad \varepsilon^k = \sum_{j \in J_k} \lambda_j^k c_j^k, \quad d^k = -g^k, \quad \xi^k = -\|g^k\|^2 - \varepsilon^k$$

**while** *not converged* **do**

Set $q_{k+1} = \exp_{p_k}(d^k)$ and compute $X_{q_{k+1}} \in \partial f(q_{k+1})$

If $f(q_{k+1}) \leq f(p_k) + m\xi^k$, then set $p_{k+1} = q_{k+1}$, else set $p_{k+1} = p_k$

Update the index set to $J_{k+1}$, set $k \leftarrow k+1$, and set

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# Convergence

- In the Euclidean case we have Geiger and Kanzow 2002, Theorem 6.80

**Theorem**
*Let the solution set* $S = \{x_* \in \mathbb{R}^n \mid f(x_*) = \inf_{x \in \mathbb{R}^n} f(x)\}$ *of the minimization problem be nonempty. Then every sequence* $\{x_k\}$ *generated by the bundle method algorithm converges to a minimum of* $f$*.*

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- In the Hadamard case, we obtain an analogous result
  - achieved by enlarging the $\varepsilon-$subdifferential
- Numerically, given $\mathrm{tol} > 0$, we employ the following stopping criterion

$$-\xi^k \leq \mathrm{tol}$$

# Implementation

The algorithm is implemented[1] in Julia using `Manopt.jl` (Bergmann 2022) which uses manifolds from `Manifolds.jl` (Axen et al. 2021). A solver call just looks like

```
p_star = bundle_method(M, f, ∂f, p0)
```

where

- `M` is a Hadamard manifold
- `f` is the objective function
- `∂f` is a subgradient of the objective function
- `p0` is an initial point on the manifold
- the parameter for the descent test is set at a default $m = 0.0125$

[1] not yet in a release version

# Numerical Example: $\mathcal{H}^4$ and $\mathcal{P}(3)$

Let $\mathcal{M}_1 = \mathcal{H}^4$ be the four-dimensional hyperbolic space, and let $\mathcal{M}_2 = \mathcal{P}(3)$ be the space of $3 \times 3$ symmetric positive definite matrices. Let $q_1, \ldots, q_n \in \mathcal{M}_j$ be $n = 100$ random data points for $j \in \{1, 2\}$.

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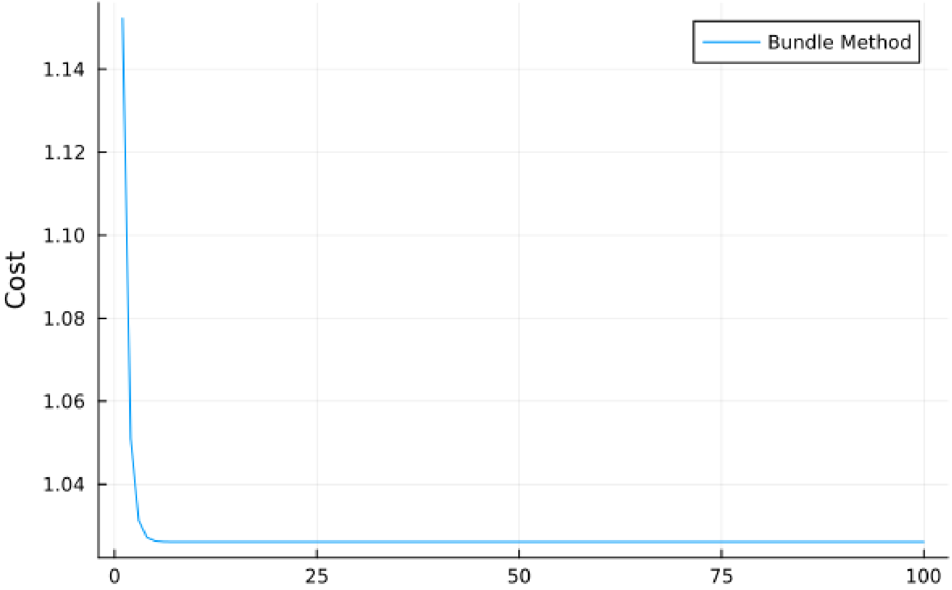
for $i, j \in \{1, 2\}$. We want to solve

$$\operatorname*{arg\,min}_{p \in \mathcal{M}_j} g_j(p)$$

for $j \in \{1, 2\}$.

# Numerical Example: $\mathcal{H}^4$



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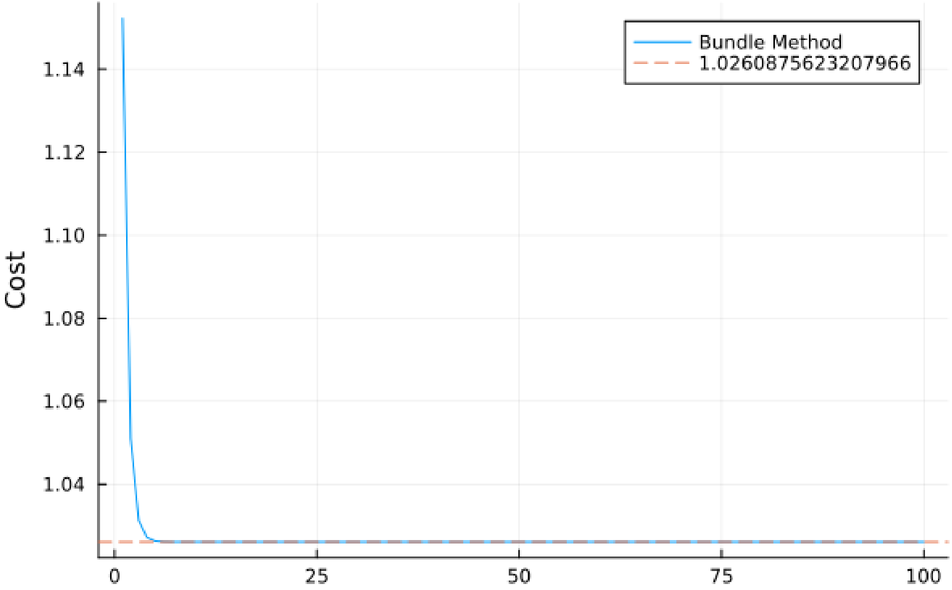

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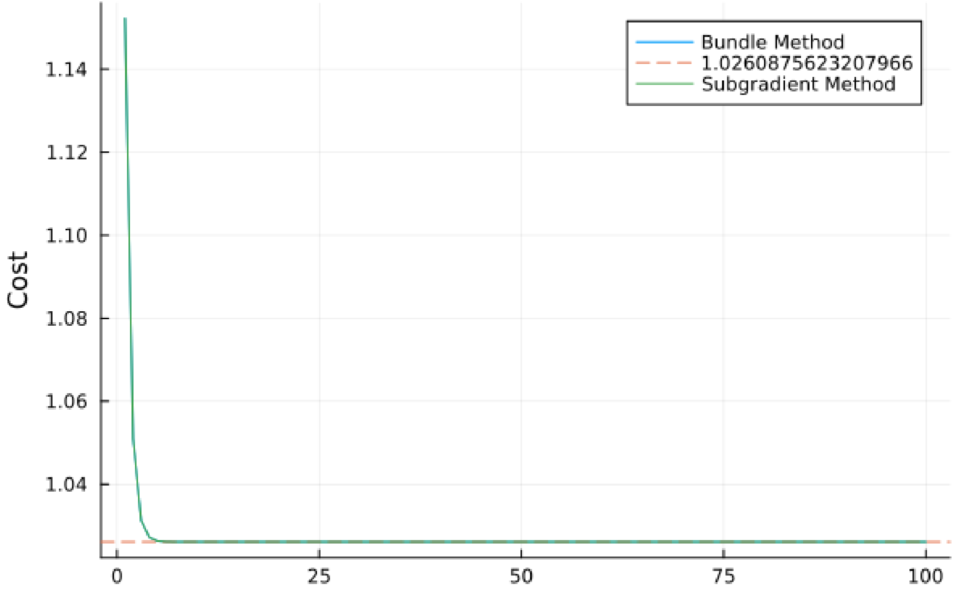

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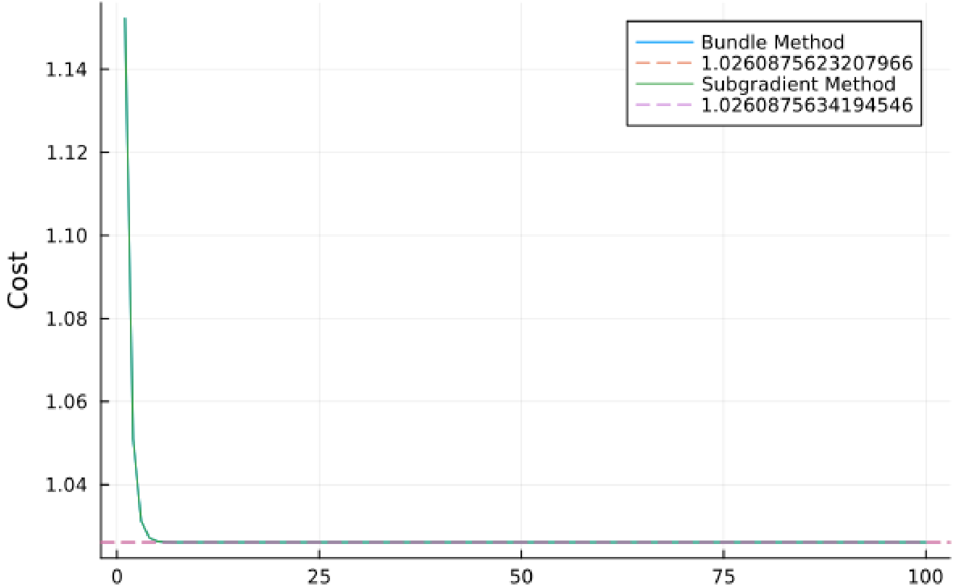

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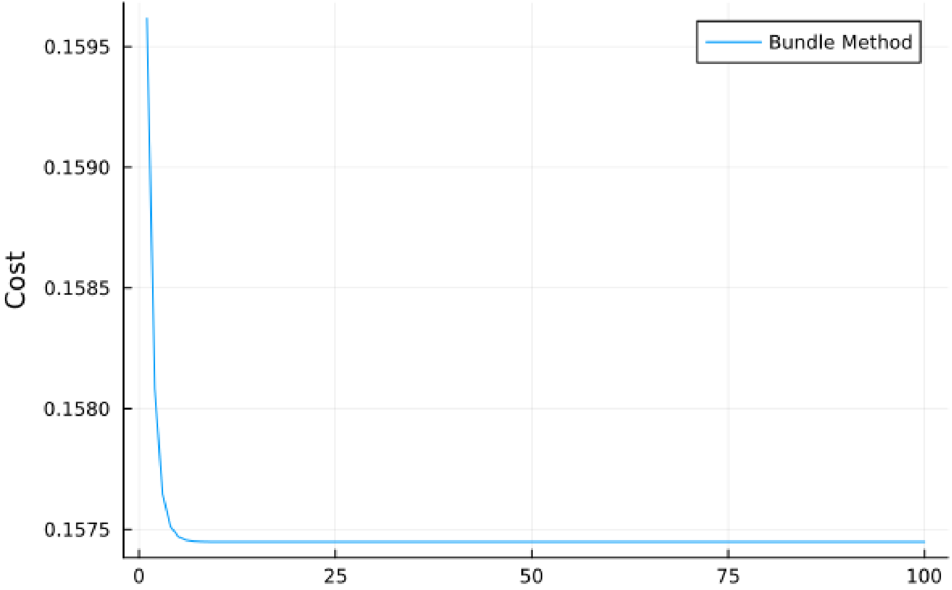

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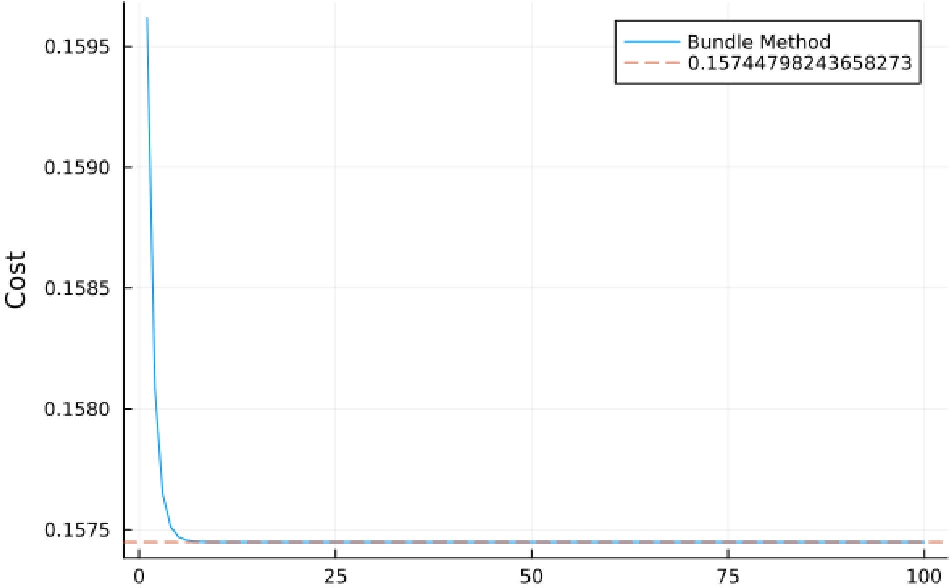

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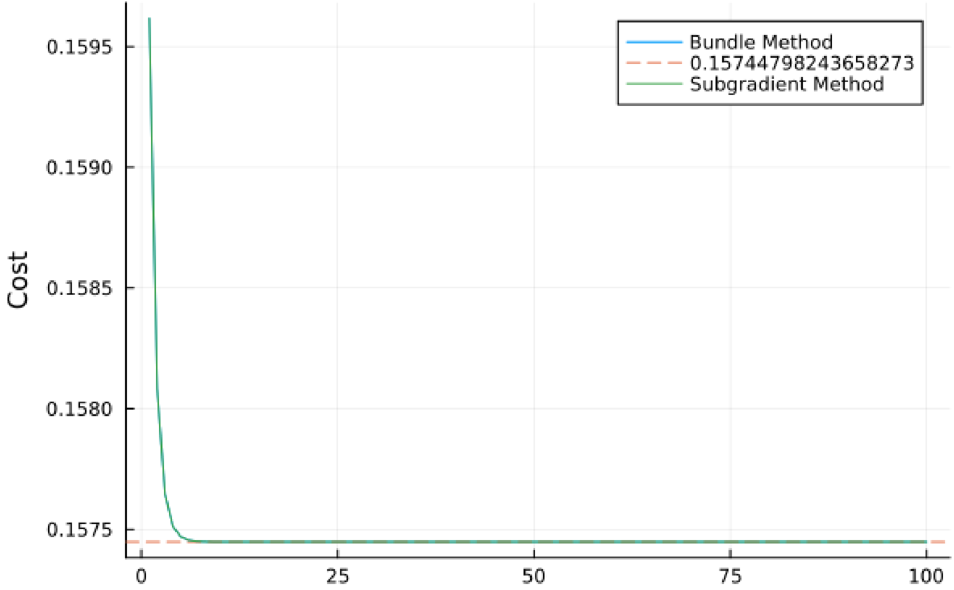

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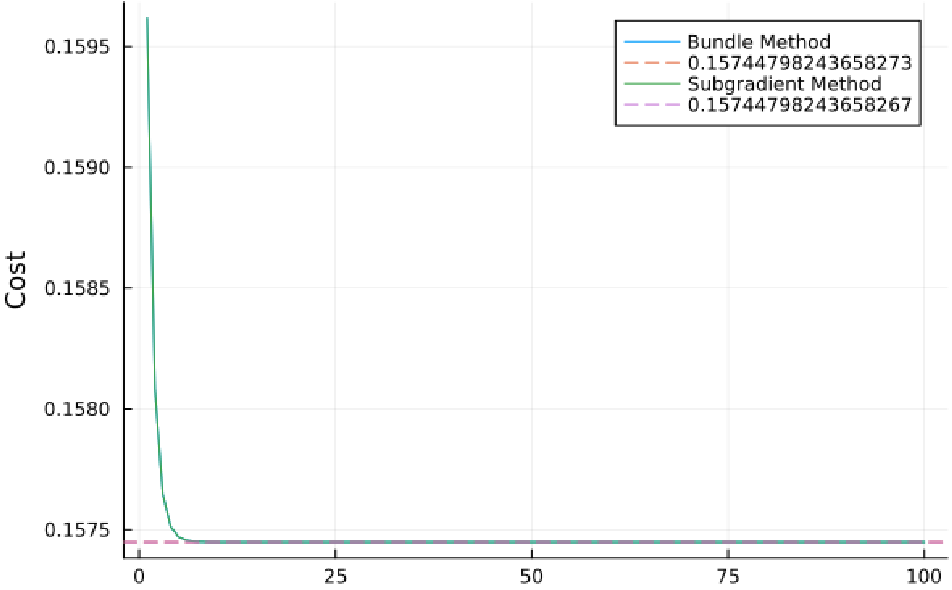

# Summary and Future Work

What we did:

- presented the bundle method for geodesically convex functions on Hadamard manifolds
- touched upon convergence
- showed two numerical examples

To do:

- further investigate convergence
- generic Riemannian manifolds

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