Nonsmooth Optimization on Riemannian Manifolds

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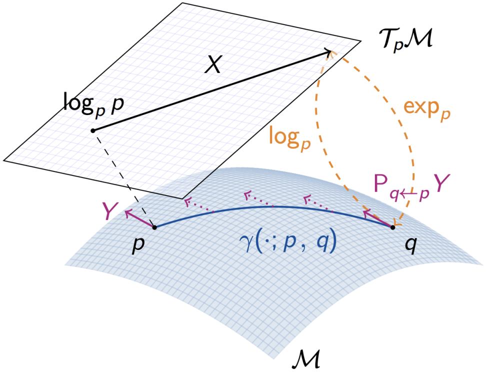
Riemannian Geometry

A d-dimensional Riemannian manifold can be defined informally as a set $\mathcal{M}$ covered with a suitable collection of charts that identify subsets of $\mathcal{M}$ with open subsets of $\mathbb{R}^d$, paired with a continuously varying inner product on the tangent spaces $\mathcal{T}_p\mathcal{M}$. [Absil, Mahony, and Sepulchre 2008]

Riemannian Geometry

- Geodesic
  $\gamma_{pq}\colon [0,1] \to \mathcal{M}$ with initial velocity $\dot\gamma_{pq}(0) = X_p \in \mathcal{T}_p\mathcal{M}$
- Inner product
  $(\cdot,\cdot)_p : \mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \to \mathbb{R}$
- Exponential map
  $\exp_p X_p = \gamma_{pq}(1) = q$
- Logarithmic map
  $\log_p q = \exp_p^{-1} q = X_p$
- Parallel transport
  $\mathrm{P}_{q\leftarrow p}\colon \mathcal{T}_p\mathcal{M} \to \mathcal{T}_q\mathcal{M}$
- Sectional curvature κ



The Problem

Goal Find numerically and fast

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} f(p),$$

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- find a “good replacement” for the $+$ operation
- no “global space of directions”: $T_p\mathcal{M} \neq T_q\mathcal{M}$

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Task How to handle the case when f is nonsmooth?

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Example application. Given $A, B \in \mathbb{R}^{n\times d}$, $\mathcal{M} = \mathrm{SO}(d)$

- orthogonal Procrustes: $f(p) = \lVert A - Bp\rVert_{\mathrm{F}}$ has a closed form solution.
- spectral Procrustes: $f(p) = \lVert A - Bp\rVert_2$ does not

Geodesic Convexity

- A set $\mathcal{C} \subseteq \mathcal{M}$ is called strongly geodesically convex if for all $p, q \in \mathcal{C}$ the geodesic $\gamma_{pq} \colon [0,1] \to \mathcal{M}$ is unique and lies in $\mathcal{C}$.

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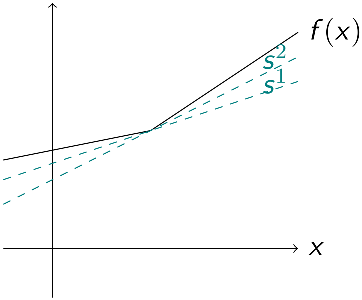
- A function $f \colon \mathcal{C} \to \overline{\mathbb{R}}$ is called geodesically convex if for all $p, q \in \mathcal{C}$ the composition $f \circ \gamma_{pq}(t)$ is convex in the usual sense.

The Convex Subdifferential(s)

For a convex function, the subdifferential is defined as

$$\partial f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + (s)^T(y - x) \text{ for all } y \in \mathbb{R}^n \right\}$$

and it is a non-empty, closed and convex subset.



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Let $\varepsilon > 0$. The $\varepsilon-$subdifferential is

$$\partial_\varepsilon f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + (s)^T (y - x) - \varepsilon \text{ for all } y \in \mathbb{R}^n \right\}$$

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$$\partial f(p) = \left\{ X_p \in \mathcal{T}_p\mathcal{M} \,\middle|\, f(q) \geq f(p) + \left(X_p, \log_p q\right) \text{ for all } q \in \operatorname{dom} f \right\}$$

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The Riemannian Convex Bundle Method

joint work with

Ronny Bergmann and Roland Herzog

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- $f\colon \mathcal{M} \to \overline{\mathbb{R}}$ is such that
  - $\operatorname{dom} f \neq \emptyset$ strongly geodesically convex
  - $\operatorname{diam}(\operatorname{dom} f) < \infty$ if $\kappa \neq 0$
  - $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ in $\mathcal{M}$
  - geodesically convex
  - lower semi-continuous: $\liminf_{q \to p} f(q) \geq f(p)$

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Goal Solve this optimization problem with a convex bundle method.

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Goal Approximate ∂f with $\partial_\varepsilon f$ on $\operatorname{int}(\operatorname{dom} f)$.

Approximating the $\varepsilon-$subdifferential

Given $x^{(0)}, \ldots, x^{(k)} \in \mathbb{R}^n$, and $s^{(j)} \in \partial f(x^{(j)})$ for $j = 0, \ldots, k$, and

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$$s^{(j)} \in \partial_{e_j^{(k)}} f(x^{(k)}).$$

This gives an inner approximation of $\partial_\varepsilon f(x^{(k)})$ as:

$$G_\varepsilon^{(k)} := \left\{ \sum_{j=0}^{k} \lambda_j s^{(j)} \,\middle|\, \sum_{j=0}^{k} \lambda_j \, e_j^{(k)} \leq \varepsilon, \ \lambda \in \Delta_k \right\} \subseteq \partial_\varepsilon f(x^{(k)})$$

with $\Delta_k := \{\lambda \in \mathbb{R}^{k+1} \mid \sum_{j=0}^{k} \lambda_j = 1, \ \lambda_j \geq 0, \ j = 0, \dots, k\}$.

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Main challenge on manifolds: given $p^{(0)}, \dots, p^{(k)} \in \mathcal{M}$, and $X_{p^{(j)}} \in \partial f(p^{(j)})$, then

$$\mathrm{P}_{p^{(k)} \leftarrow p^{(j)}} X_{p^{(j)}} \in \partial_c f(p^{(k)}) \quad \text{for some} \quad c > 0?$$

Curvature Correction

Let $\delta := \operatorname{diam}(\operatorname{dom} f)$ and

$$\zeta_{1,\omega}(\delta) := \begin{cases} 1 & \text{if } \omega \geq 0, \\ \sqrt{-\omega}\,\delta \coth(\sqrt{-\omega}\,\delta) & \text{if } \omega < 0, \end{cases}$$

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[Alimisis et al. 2021, Appendix C]

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With

$$e_j^{(k)} := f(p^{(k)}) - f(p^{(j)}) - \left(X_{p^{(j)}}, \log_{p^{(j)}} p^{(k)} \right),$$

$$r_j^{(k)} := \varrho \|\log_{p^{(j)}} p^{(k)}\| \|X_{p^{(j)}}\|,$$

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for $X_{p_j} \in \partial f(p^{(j)})$, we get

$$G_\varepsilon^k := \left\{ \sum_{j=1}^{k} \lambda_j \mathrm{P}_{p^{(k)} \leftarrow p^{(j)}} X_{p^{(j)}} \;\middle|\; \sum_{j=1}^{k} \lambda_j \left(e_j^{(k)} + r_j^{(k)}\right) \leq \varepsilon,\ \lambda \in \Delta_k \right\} \subseteq \partial_\varepsilon f(p^{(k)})$$

[Bergmann, Herzog, and HJ 2024]

NTNU

The Euclidean Subproblem

Let $k \in \mathbb{N}$ and $j \in \{0, \ldots, k\} =: J^{(k)}$ and let λ_j be convex coefficients.

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where λ_j are solution to

$$\begin{aligned} \operatorname*{arg\,min}_{\lambda \in \mathbb{R}^{|J^{(k)}|}} \quad & \frac{1}{2} \left\| \sum_{j \in J^{(k)}} \lambda_j s^{(j)} \right\|^2 + \sum_{j \in J^{(k)}} \lambda_j e_j^{(k)} \\ \text{s. t.} \quad & \sum_{j \in J^{(k)}} \lambda_j = 1, \quad \lambda_j \geq 0 \qquad \text{for all } j \in J^{(k)} \end{aligned}$$

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$$X_{q^{(j)}} \in \partial f(q^{(j)}) \implies \sum_{j=0}^{k} \lambda_j \mathrm{P}_{p^{(k)} \leftarrow q^{(j)}} X_{q^{(j)}} \in \partial_\varepsilon f(p^{(k)}) \quad \text{iff} \quad \varepsilon \geq \sum_{j=0}^{k} \lambda_j \left(e_j^{(k)} + r_j^{(k)} \right)$$

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$$\text{s. t.} \quad \sum_{j \in J^{(k)}} \lambda_j = 1, \quad \lambda_j \geq 0 \qquad \text{for all } j \in J^{(k)}$$

The Euclidean Convex Bundle Method

Data: $x^{(0)} = y^{(0)} \in \mathbb{R}^n$, $g^{(0)} = s^{(0)} \in \partial f(x^{(0)})$, $m \in (0,1)$,
$\varepsilon^{(0)} = e_0^{(0)} = 0$, $k = 0$, $J^{(k)} = \{0\}$

while *not converged* **do**

Compute a solution $\lambda \in \mathbb{R}^{|J^{(k)}|}$ of the subproblem and set

$$g^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} s^{(j)}, \quad \varepsilon^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} e_j^{(k)},$$

$$d^{(k)} := -g^{(k)}, \quad \xi^{(k)} := -\|g^{(k)}\|^2 - \varepsilon^{(k)}$$

Set $y^{(k+1)} := x^{(k)} + d^{(k)}$.

If $f(y^{(k+1)}) \leq f(x^{(k)}) + m\xi^{(k)}$, $x^{(k+1)} := y^{(k+1)}$, else $x^{(k+1)} := x^{(k)}$.

Compute $s^{(k+1)} \in \partial f(y^{(k+1)})$, update $J^{(k+1)}$, and $e_j^{(k+1)}$

end

Result: $x^{(k_*)}$, for some $k_* \in \mathbb{N}$.

The Riemannian Convex Bundle Method

Data: $p^{(0)} = q^{(0)} \in \operatorname{int}(\operatorname{dom} f)$, $g^{(0)} = X_1 \in \partial f(p^{(0)})$, $m \in (0,1)$,
$\varepsilon^{(0)} = c_0^{(0)} = 0$, $k = 0$, $J^{(k)} = \{0\}$, $\beta > 0$, $\varrho \in \mathbb{R}$,

while *not converged* **do**

Compute a solution $\lambda \in \mathbb{R}^{|J^{(k)}|}$ of the subproblem and set

$$g^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} \mathrm{P}_{p^{(k)} \leftarrow q^{(j)}} X_{q^{(j)}}, \quad \varepsilon^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} \left(e_j^{(k)} + r_j^{(k)} \right),$$

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Set $q^{(k+1)} := \exp_{p^{(k)}}(t^{(k)} d^{(k)})$ [1]

If $f(q^{(k+1)}) \le f(p^{(k)}) + m\xi^{(k)}$, $p^{(k+1)} := q^{(k+1)}$, else $p^{(k+1)} := p^{(k)}$ [2] .

Compute $X_{q^{(k+1)}} \in \partial f(q^{(k+1)})$, update $J^{(k+1)}$, and $e_j^{(k+1)}$ and $r_j^{(k+1)}$

end

Result: $p^{(k_*)}$, for some $k_* \in \mathbb{N}$.

---

[1] Backtrack $t^{(k)} \leftarrow \beta t^{(k)}$ if $q^{(k+1)} \notin \operatorname{int}(\operatorname{dom} f)$ or equal to $p^{(k)}$

[2] Look for a $\bar{t}^{(k)} \le t^{(k)}$ that satisfies a geometric condition

Convergence

- In the Euclidean case [Geiger and Kanzow 2002, Theorem 6.80] holds.

Theorem

Let the solution set $S = \{x^* \in \mathbb{R}^n \mid f(x^*) = \inf_{x \in \mathbb{R}^n} f(x)\}$ *of the minimization problem be nonempty. Then every sequence* $\{x^{(k)}\}$ *generated by the bundle method algorithm converges to a minimizer of* f*.*

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- In the Riemannian case, assuming no accumulation point of $p^{(k)}$ is allowed to lie on $\partial \operatorname{dom} f$, we have an analogous result.

[Bergmann, Herzog, and HJ 2024]

Need to backtrack until

$$\left(\mathrm{P}_{p^{(k)}\leftarrow\bar{q}^{(k+1)}} X_{\bar{q}^{(k+1)}}\,,\ \bar{t}^{(k)}\, g^{(k)}\right) < -m\, \bar{t}^{(k)}\, \xi^{(k)} - e^{(k)}_{k+1} - r^{(k)}_{k+1}, \qquad (1)$$

as inspired by [Bagirov et al. 2020]

Implementation

The algorithm is implemented in Julia using `Manopt.jl` ([Bergmann 2022]) and `Manifolds.jl` ([Axen et al. 2023]). A solver call looks like [1]

```
p* = convex_bundle_method(M, f, ∂f, p0; diameter = δ,
     domain = dom f, k_max = Ω, k_min = ω, m = 10^−3)
```

where

- `M` is a Riemannian manifold
- `f` is the objective function
- `∂f` is a subgradient of the objective function
- `p0` is an initial point on the manifold

The default stopping criterion for the algorithm is set to

$$-\xi^{(k)} \leq 10^{-8}.$$

[1] https://manoptjl.org/stable/solvers/convex_bundle_method/

The Proximal Gradient Method

joint work with

Ronny Bergmann, Paula J. John, Lukas Klingbiel, and Max Pfeffer

(preliminary results)

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where

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  - bounded sectional curvature $\omega \leq \kappa \leq \Omega$
- and
  - $g\colon \mathcal{M} \to \mathbb{R}$ is continuously differentiable and L_g-smooth
  - $h\colon \mathcal{M} \to \mathbb{R}$ is (locally) Lipschitz continuous
  - f has compact level sets
  - optimal set of Equation (2) is nonempty

The Problem

Consider the following minimization problem

$$\text{minimize } f(p) = g(p) + h(p), \quad p \in \mathcal{M}, \tag{2}$$

where

- $\mathcal{M}$ is a Riemannian manifold
  - complete
  - bounded sectional curvature $\omega \leq \kappa \leq \Omega$
- and
  - $g : \mathcal{M} \to \mathbb{R}$ is continuously differentiable and L_g-smooth
  - $h : \mathcal{M} \to \mathbb{R}$ is (locally) Lipschitz continuous
  - f has compact level sets
  - optimal set of Equation (2) is nonempty

Goal Solve this problem with the proximal gradient method.

The Proximal Operator

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a real number $\lambda > 0$, the proximal operator of f is

$$\operatorname{prox}_{\lambda f}(x) := \operatorname*{arg\,min}_{y \in \mathbb{R}^n} f(y) + \frac{1}{2\lambda} \operatorname{dist}^2(y, x),$$

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$$\operatorname*{arg\,min}_{y \in \mathbb{R}^n} g(x) + (\operatorname{grad} g(x)\,,\ y - x) + \frac{1}{2\lambda}\operatorname{dist}^2(x, y) + h(y).$$

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Taylor expand the distance term to second order and obtain

$$\operatorname*{arg\,min}_{y \in \mathbb{R}^n} g(x) + \frac{1}{2\lambda} \operatorname{dist}^2 (x - \lambda \operatorname{grad} g(x), y) + h(y).$$

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Since $g(x)$ does not change the minimizer, we get

$$\operatorname{prox}_{\lambda h}(x - \lambda \operatorname{grad} g(x)) = \operatorname*{arg\,min}_{y\in\mathbb{R}^n} \frac{1}{2\lambda}\operatorname{dist}^2(x - \lambda \operatorname{grad} g(x), y) + h(y)$$

Derivation of the Method

Main idea Replace g by a proximal regularization of its linearized function. For $\lambda > 0$ and a point p, compute a new candidate as

$$\operatorname*{arg\,min}_{q \in \mathcal{M}} g(p) + \left(\operatorname{grad} g(p)\,,\ \log_p q \right) + \frac{1}{2\lambda} \operatorname{dist}^2(p, q) + h(q).$$

Taylor expand the distance term to second order and obtain

$$\operatorname*{arg\,min}_{q \in \mathcal{M}} g(p) + \frac{1}{2\lambda} \operatorname{dist}^2 \left(\exp_p(-\lambda \operatorname{grad} g(p)), q \right) + h(q).$$

Since $g(p)$ does not change the minimizer, we get

$$\operatorname{prox}_{\lambda h}(\exp_p(-\lambda \operatorname{grad} g(p))) = \operatorname*{arg\,min}_{q \in \mathcal{M}} \frac{1}{2\lambda} \operatorname{dist}^2(\exp_p(-\lambda \operatorname{grad} g(p)), q) + h(q)$$

Proximal Gradient Method

Data: g, $\operatorname{grad} g$, h, $\operatorname{prox}_{\lambda h}$, a sequence $\lambda^{(k)}$, an initial point $x^{(0)} \in \mathbb{R}^n$.

while *convergence criterion is not fulfilled* **do**

$\quad x^{(k+1)} = \operatorname{prox}_{\lambda^{(k)} h}\left(x^{(k)} - \lambda^{(k)} \operatorname{grad} g(x^{(k)})\right)$

$\quad$ Set $k := k + 1$

end

Result: $x^{(k_*)}$, for some $k_* \in \mathbb{N}$.

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Convergence

In the Euclidean case we need a backtracking strategy given by three parameters $s > 0$, $\gamma \in (0,1)$, $\eta > 1$ such that

$$f(x^{(k)}) - f(\mathrm{prox}_{s\eta^{i_k} h}(x^{(k)})) \geq \gamma s \eta^{i_k} \,\mathrm{dist}^2(x^{(k)}, \mathrm{prox}_{s\eta^{i_k} h}(x^{(k)})), \tag{3}$$

$i_k \geq 0$ is the smallest integer for which the condition is satisfied.

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Theorem

Given either a constant step-size $\lambda \in \left(0, \frac{2}{L_g}\right)$ or one that is chosen by the backtracking procedure Equation (3), then

1. *the sequence $\left(f(p^{(k)})\right)$ is nonincreasing, and $f(x^{(k+1)}) < f(x^{(k)})$ if and only if $x^{(k)}$ is not a stationary point of the original problem;*
2. $\lambda \operatorname{dist}(x^{(k)}, x^{(k+1)}) \to 0$ *as* $k \to \infty$*;*
3. $\min_{n=0,1,\dots,k} \lambda^{(n)} \operatorname{dist}(x^{(n)}, x^{(n+1)}) = O\left(\frac{1}{\sqrt{k+1}}\right)$*;*
4. *all limit points of the sequence $(x^{(k)})$ are stationary points of the original problem.*

[Beck 2017, Theorem 10.15]

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Key differences:

- if $\Omega > 0$, no global law of cosines
  - need to ensure $\lambda \in \left(0, \frac{\zeta_{2,\Omega}(r)}{L_g}\right)$ and $\zeta_{2,\Omega}(r) > 0$, where

$$r \leq \operatorname{dist}(p^{(k)}, q^{(k)}) + \operatorname{dist}(q^{(k)}, p^{(k+1)}),$$

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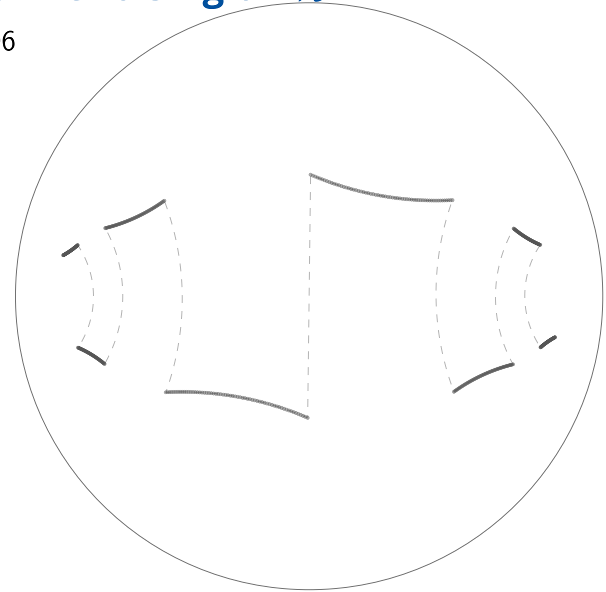
- working with locally Lipschitz functions (not necessarily geodesically convex)
  - we can achieve ε-statioinarity.

Numerical Example: Signal Denoising on $\mathcal{H}^2$

- signal $q \in \mathcal{M} = (\mathcal{H}^2)^n$, $n = 496$



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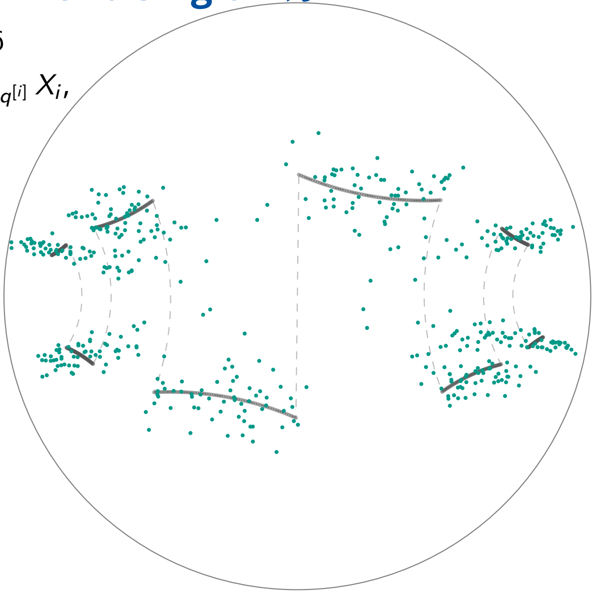

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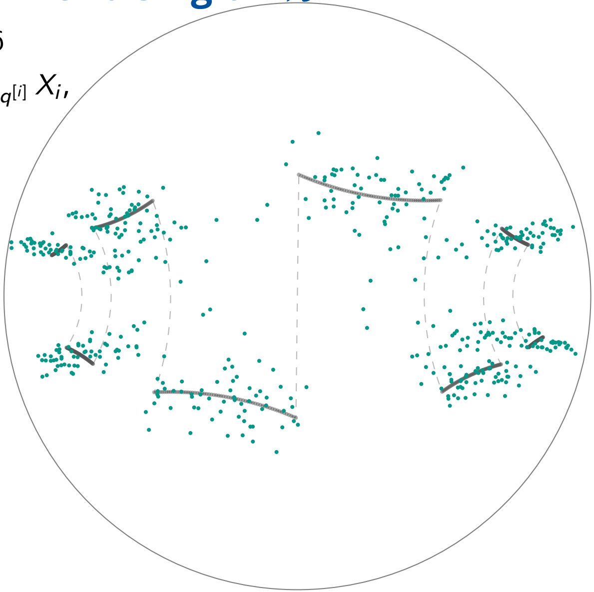
- ROF Model:

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} \frac{1}{n} d_{\mathcal{M}}(p, \bar{q})^2 + \alpha \sum_{i=1}^{n-1} d_{\mathcal{H}^2}(p^{[i]}, p^{[i+1]})$$



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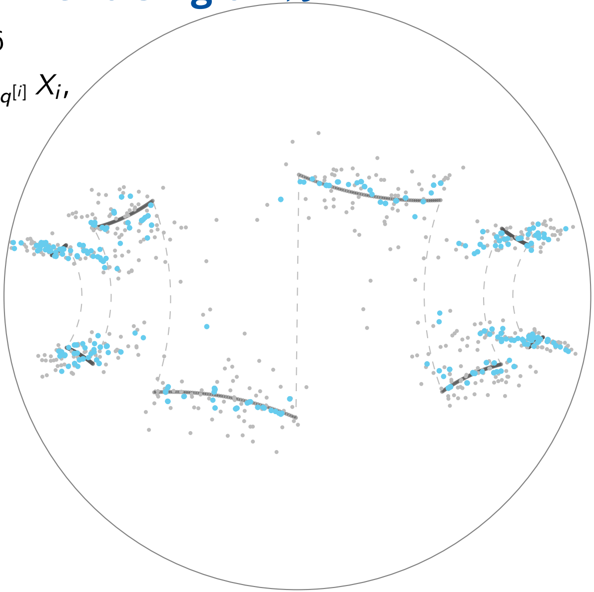
- Setting $\alpha = 0.5$ yields RCBM reconstruction p_1^*



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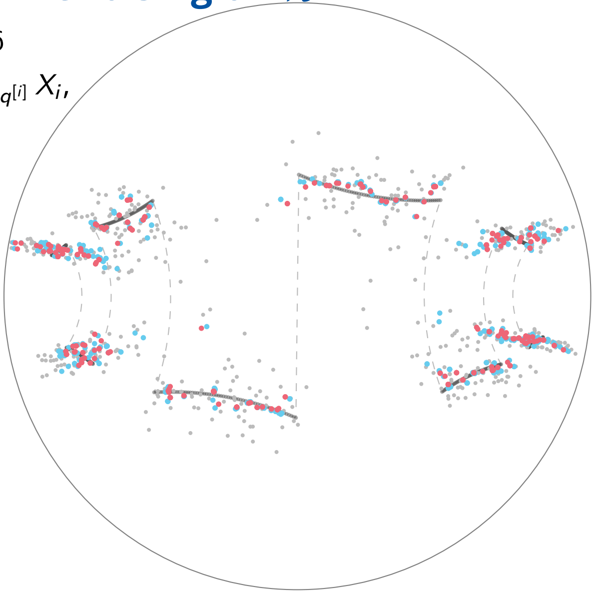
- and PGM reconstruction p_2^*



Signal Denoising - Algorithms[2]

- Riemannian Convex Bundle Method (RCBM) [Bergmann, Herzog, and HJ 2024]
- Proximal Bundle Algorithm (PBA) [Hoseini Monjezi, Nobakhtian, and Pouryayevali 2021]
- Subgradient Method (SGM) [Ferreira and Oliveira 1998]
- Cyclic Proximal Point Algorithm (CPPA) [Bačák 2014]
- Proximal Gradient Method (PGM) - work in progress

Algorithm	Iter.	Time (sec.)	Objective	Error
RCBM	5000	13.892	1.4023×10^{-1}	1.3692×10^{-2}
PBA	5000	9.519	1.4289×10^{-1}	1.3032×10^{-2}
SGM	5000	7.897	1.4622×10^{-1}	1.2460×10^{-2}
CPPA	5000	3.739	1.3191×10^{-1}	1.7361×10^{-2}
PGM	5000	3.555	1.3191×10^{-1}	1.7351×10^{-2}

[2] The code for the experiment is available at `juliamanifolds.github.io/ManoptExamples.jl/stable/examples/H2-Signal-TV/`

Numerical Example: Riemannian Median on $\mathcal{S}^d$

- $\mathcal{S}^d$ $d-$dimensional sphere
- $\bar{p}$ north pole
- $q^{(1)}, \ldots, q^{(n)} \in \mathcal{S}^d$ are $n = 1000$ Gaussian random data points in $B_{\frac{\pi}{6}}(\bar{p})$
- $\mathcal{D} = \left\{ q^{(1)}, \ldots, q^{(n)} \,\middle|\, q^{(j)} \in \mathcal{S}^d \text{ for all } j = 1, \ldots, n \right\}$

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- $\mathcal{D} = \{q^{(1)}, \dots, q^{(n)} \mid q^{(j)} \in \mathcal{S}^d \text{ for all } j = 1, \dots, n\}$
- $f(p) = \begin{cases} \frac{1}{n} \sum_{j=1}^{n} \operatorname{dist}(p, q^{(j)}) & \text{if } p \in B_{\frac{\pi}{6}}(\bar{p}), \\ +\infty & \text{otherwise.} \end{cases}$

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Solve

$$p^* := \arg\min_{p \in \mathcal{S}^d} f(p)$$

Riemannian Median on $\mathcal{S}^d$ - Algorithms[3]

	RCBM			PBA		
Dimension	Iter.	Time (sec.)	Objective	Iter.	Time (sec.)	Objective
2	43	1.58×10^{-2}	0.258 90	71	1.84×10^{-2}	0.258 90
4	74	2.30×10^{-2}	0.253 53	62	1.68×10^{-2}	0.253 53
32	102	4.30×10^{-2}	0.259 89	64	2.73×10^{-2}	0.259 89
1024	103	8.90×10^{-1}	0.266 99	68	6.23×10^{-1}	0.266 99
32 768	80	2.72×10^{1}	0.259 30	65	2.92×10^{1}	0.259 30

	SGM		
Dimension	Iter.	Time (sec.)	Objective
2	401	9.70×10^{-2}	0.258 90
4	5000	1.33	0.253 53
32	231	9.63×10^{-2}	0.259 89
1024	185	1.99	0.266 99
32 768	157	2.05×10^{2}	0.259 30

[3] The code for the experiment is available at `juliamanifolds.github.io/ManoptExamples.jl/stable/examples/RCBM-Median/`

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Conclusion and Future Work

In summary:

- introduced tools for nonsmooth optimization on Riemannian manifolds
- introduced the Riemannian Convex Bundle Method
- discussed convergence and related challenges
- introduced a preliminary version of a Riemannian Proximal Gradient Method
- showed two numerical examples

To do:

- work on the convergence analysis for the Riemannian Proximal Gradient Method
- acceleration?

Thank you!

Selected References

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