Hajg Jasa.

- Studied both Physics and Mathematics at University of Milano-Bicocca
- main focus: Differential Geometry
- Reserach Internship at University of Zurich: statistical modelling and data analysis
- Since August 2022: PhD student at IMF

Thesis Project.
Nonsmooth Optimization on Riemannian manifolds

Goal, Find (numerically / fast)

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} f(p),$$

where $\mathcal{M}$ is a Riemannian manifold.

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- Find is a "good replacement" $+$
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Example application. Given $A, B \in \mathbb{R}^{d\times n}$, $\mathcal{M} = \mathrm{SO}(d)$

- orthogonal Procrustes: $f(p) = \|A - pB\|_{\mathrm{F}}$ has a closed form solution.
- spectral Procrustes: $f(p) = \|A - pB\|_2$ does not

The Riemannian Convex Bundle Method

Hajg Jasa

joint work with
Ronny Bergmann and Roland Herzog

DNA Seminar, NTNU, Trondheim

April 19th, 2024

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  - $\operatorname{dom} f \neq \emptyset$, $\operatorname{diam}(\operatorname{dom} f) < \infty$ if $\kappa \neq 0$, strongly geodesically convex set and $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ in $\mathcal{M}$
  - geodesically convex: $f \circ \gamma$ convex
  - lower semi-continuous: $\liminf_{q \to p} f(q) \geq f(p)$

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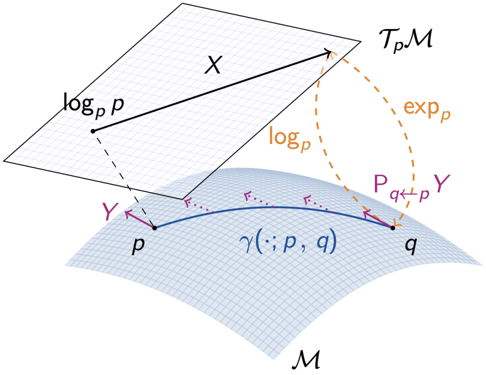
Goal. Solve this optimization problem with a convex bundle method.

Riemannian Geometry

Notation

- Smooth Riemannian manifold $\mathcal{M}$
- Tangent space $\mathcal{T}_p\mathcal{M}$ at the point $p \in \mathcal{M}$
- Inner product $(\cdot,\cdot)_p : \mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \to \mathbb{R}$
- Exponential map $\exp_p X_p = \gamma_{pq}(1) = q$
- Logarithmic map $\log_p q = \exp_p^{-1} q = X_p$
- Parallel transport $\mathrm{P}_{q\leftarrow p} : \mathcal{T}_p\mathcal{M} \to \mathcal{T}_q\mathcal{M}$
- Sectional curvature κ

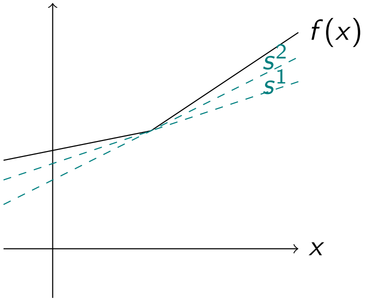


The Convex Subdifferential(s)

For a convex function, the subdifferential is defined as

$$\partial f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + (s)^T (y - x) \text{ for all } y \in \mathbb{R}^n \right\}$$

and it is a non-empty, closed and convex subset.



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Let $\varepsilon > 0$. The $\varepsilon-$subdifferential is

$$\partial_\varepsilon f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + (s)^T (y - x) - \varepsilon \text{ for all } y \in \mathbb{R}^n \right\}$$

and

$$\partial f(x) \subseteq \partial_\varepsilon f(x)$$

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- generates sequences of *candidate* points and *stability centers*

Goal. Approximate ∂f with $\partial_\varepsilon f$ on $\operatorname{int}(\operatorname{dom} f)$.

Approximating the $\varepsilon-$subdifferential

Given $x^{(0)}, \ldots, x^{(k)} \in \mathbb{R}^n$, and $s^{(j)} \in \partial f(x^{(j)})$ for $j = 0, \ldots, k$, and

Approximating the $\varepsilon-$subdifferential

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then

$$s^{(j)} \in \partial_{e_j^{(k)}} f(x^{(k)}).$$

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Characterize an inner approximation of $\partial_\varepsilon f(p)$ as:

$$G_\varepsilon^{(k)} := \left\{ \sum_{j=0}^{k} \lambda_j s^{(j)} \,\middle|\, \sum_{j=0}^{k} \lambda_j\, e_j^{(k)} \leq \varepsilon,\ \sum_{j=0}^{k} \lambda_j = 1,\ \lambda_j \geq 0\, \forall j = 0, \dots, k \right\}$$

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with $G_\varepsilon^{(k)} \subseteq \partial_\varepsilon f(x^{(k)})$.

Main challenge on manifolds: given $p^{(0)}, \ldots, p^{(k)} \in \mathcal{M}$, and $X_{p^{(j)}} \in \partial f(p^{(j)})$, then

$$\mathrm{P}_{p^{(k)} \leftarrow p^{(j)}} X_{p^{(j)}} \in \partial_c f(p^{(k)}) \quad \text{for some} \quad c > 0?$$

Curvature Correction

Using the upper bound Ω on the curvature, define

$$c_j^{(k)} := f(p^{(k)}) - f(p^{(j)}) - \left(X_{p^{(j)}}, \log_{p^{(j)}} p^{(k)}\right) \quad \text{if } \Omega \leq 0,$$
$$c_j^{(k)} := f(p^{(k)}) - f(p^{(j)}) + \|X_{p^{(j)}}\| \|\log_{p^{(j)}} p^{(k)}\| \quad \text{if } \Omega > 0.$$

[Bergmann, Herzog, and HJ 2024].

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with $G_\varepsilon^{(k)} \subseteq \partial_\varepsilon f(p^{(k)})$, and $\mathrm{P}_{p^{(k)} \leftarrow p^{(j)}} X_{p^{(j)}} \in \partial_{c_j^{(k)}} f(p^{(k)})$.

The Euclidean Subproblem

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$$\operatorname*{arg\,min}_{\lambda \in \mathbb{R}^{|J^{(k)}|}} \quad \frac{1}{2} \left\| \sum_{j \in J^{(k)}} \lambda_j s^{(j)} \right\|^2 + \sum_{j \in J^{(k)}} \lambda_j e_j^{(k)}$$

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The Euclidean Convex Bundle Method

Data: $x^{(0)} = y^{(0)} \in \mathbb{R}^n$, $g^{(0)} = s^{(0)} \in \partial f(x^{(0)})$, $m \in (0,1)$,
$\varepsilon^{(0)} = e_0^{(0)} = 0$, $k = 0$, $J^{(k)} = \{0\}$

while *not converged* **do**

Compute a solution $\lambda^{(k)} \in \mathbb{R}^{|J^{(k)}|}$ of the subproblem and set

$$g^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} s^{(j)}, \quad \varepsilon^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} e_j^{(k)},$$

$$d^{(k)} := -g^{(k)}, \quad \xi^{(k)} := -\|g^{(k)}\|^2 - \varepsilon^{(k)}$$

Set $y^{(k+1)} := x^{(k)} + d^{(k)}$.

If $f(y^{(k+1)}) \leq f(x^{(k)}) + m\xi^{(k)}$, then set $x^{(k+1)} := y^{(k+1)}$, else set $x^{(k+1)} := x^{(k)}$.

Compute $s^{(k+1)} \in \partial f(y^{(k+1)})$, update $J^{(k+1)}$, and set $e_j^{(k+1)} := f(x^{(k+1)}) - f(y^{(j)}) - (s^{(j)})^T (x^{(k+1)} - y^{(j)})$ for all $j \in J^{(k+1)}$

end

Result: $x^{(k_*)}$, for some $k_* \in \mathbb{N}$.

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Data: $p^{(0)} = q^{(0)} \in \operatorname{int}(\operatorname{dom} f)$, $g^{(0)} = X_1 \in \partial f(p^{(0)})$, $m \in (0,1)$,
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Set $t^{(k)} := 1$. While $q^{(k+1)} := \exp_{p^{(k)}}(t^{(k)} d^{(k)}) \notin \operatorname{int}(\operatorname{dom} f)$ or $\operatorname{dist}(q^{(k+1)}, p^{(k)}) < t^{(k)} \|d^{(k)}\|$ backtrack $t^{(k)} = \beta t^{(k)}$.

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Convergence

- In the Euclidean case [Geiger and Kanzow 2002, Theorem 6.80] holds.

Theorem
Let the solution set $S = \{x^* \in \mathbb{R}^n \mid f(x^*) = \inf_{x \in \mathbb{R}^n} f(x)\}$ *of the minimization problem be nonempty. Then every sequence* $\{x^{(k)}\}$ *generated by the bundle method algorithm converges to a minimizer of* f.

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- In the Hadamard case, assuming
  1. $t^{(k)} > m$ for all $k \geq k_*$, if a finite number of serious steps k_* occur
  2. no accumulation point of $p^{(k)}$ is allowed to lie on $\partial\,\mathrm{dom}\, f$

  we have an analogous result. [Bergmann, Herzog, and HJ 2024]

Convergence

In [Geiger and Kanzow 2002, Theorem 6.80], they are able to show that

$$\sum_{j=k_*}^{+\infty} \left(\|g^{(j)}\|^2 + \varepsilon^{(j)} \right)^2 < +\infty$$

where $k_* \in \mathbb{N}$ is the index that corresponds to the last serious iterate, namely

$$p^{(k)} = p^{(k_*)} \quad \text{for all } k \geq k_*.$$

This is possible because from the definition of $e_j^{(k)}$ one gets a bound

$$\left(s^{(k)}, g^{(k-1)} \right) < m \left(\|g^{(k-1)}\|^2 + \varepsilon^{(k-1)} \right) - e_k^{(k)},$$

which is still valid in the Hadamard case for $c_k^{(k)}$ with the added assumptions.

Implementation

The algorithm is implemented in Julia using `Manopt.jl` ([Bergmann 2022]) and `Manifolds.jl` ([Axen et al. 2023]). A solver call looks like [1]

```
p* = convex_bundle_method(M, f, ∂f, p0;
     diameter = δ, domain = dom f, k_max = Ω, m = 10^-3)
```

where

- `M` is a Riemannian manifold
- `f` is the objective function
- `∂f` is a subgradient of the objective function
- `p0` is an initial point on the manifold

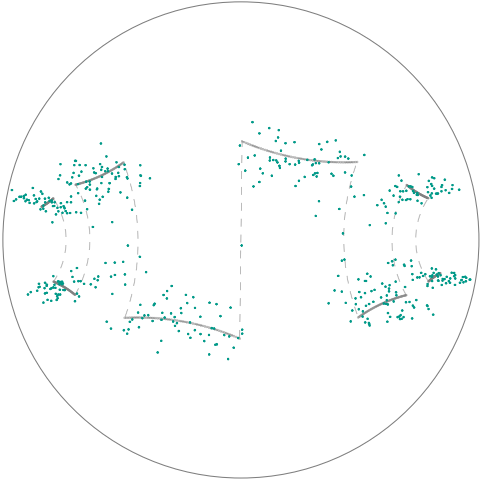
The default stopping criterion for the algorithm is set to

$$-\xi^{(k)} \leq 10^{-8}.$$

[1] https://manoptjl.org/stable/solvers/convex_bundle_method/

Numerical Example: Signal Denoising on $\mathcal{H}^2$

- $\mathcal{H}^2$ hyperbolic space, $n = 496$
- $\mathcal{M} = (\mathcal{H}^2)^n$
- artificial signal $q \in \mathcal{M}$
- noisy signal $\bar{q} \in \mathcal{M}$,
  $\bar{q}^{[i]} = \exp_{q^{[i]}} X_i$, $\sigma = 0.1$



signal q (gray)
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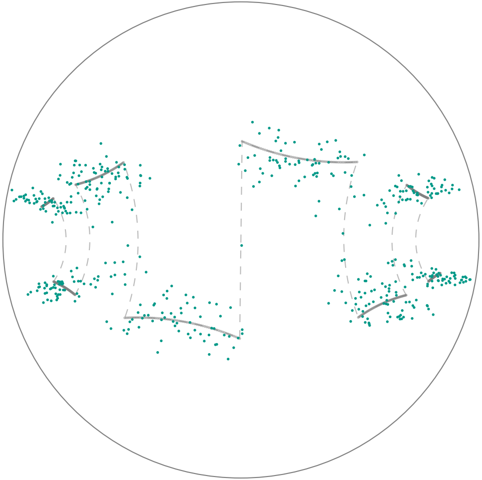
Solve the total variation problem

- $f_q(p) = \frac{1}{n}\bigl(g(p,q) + \alpha\,\mathrm{TV}(p)\bigr)$
- $g(p,q) = \frac{1}{2}\displaystyle\sum_{i=1}^{n} \operatorname{dist}(p^{[i]}, q^{[i]})^2$
- $\mathrm{TV}(p) = \displaystyle\sum_{i=1}^{n-1} \operatorname{dist}(p^{[i]}, p^{[i+1]})$
- $\alpha = 0.05$,
  $\operatorname{diam}(\operatorname{dom} f) := \texttt{floatmax()} \approx 10^{308}$



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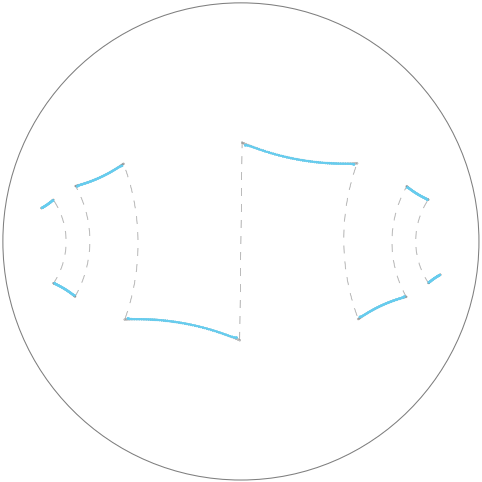
Numerical Example: Signal Denoising on $\mathcal{H}^2$

- $\mathcal{H}^2$ hyperbolic space, $n = 496$
- $\mathcal{M} = (\mathcal{H}^2)^n$
- artificial signal $q \in \mathcal{M}$
- noisy signal $\bar{q} \in \mathcal{M}$,
  $\bar{q}^{[i]} = \exp_{q^{[i]}} X_i$, $\sigma = 0.1$

Solve the total variation problem

- $f_q(p) = \frac{1}{n}\big(g(p,q) + \alpha\,\mathrm{TV}(p)\big)$
- $g(p,q) = \frac{1}{2}\sum_{i=1}^{n} \operatorname{dist}(p^{[i]}, q^{[i]})^2$
- $\mathrm{TV}(p) = \sum_{i=1}^{n-1} \operatorname{dist}(p^{[i]}, p^{[i+1]})$
- $\alpha = 0.05$,
  $\operatorname{diam}(\operatorname{dom} f) := \texttt{floatmax()} \approx 10^{308}$



signal q (gray)
reconstruction q^* (cyan).

Signal Denoising - Algorithms

- Riemannian Convex Bundle Method (RCBM) [Bergmann, Herzog, and HJ 2024]
- Proximal Bundle Algorithm (PBA) [Hoseini Monjezi, Nobakhtian, and Pouryayevali 2021]
- Subgradient Method (SGM) [Ferreira and Oliveira 1998]
- Cyclic Proximal Point Algorithm (CPPA) [Bačák 2014]

Algorithm	Iter.	Time (sec.)	Objective	Error
RCBM	3417	51.393	1.7929×10^{-3}	3.3194×10^{-4}
PBA	15 000	102.387	1.8153×10^{-3}	4.3874×10^{-4}
SGM	15 000	99.604	1.7920×10^{-3}	3.3080×10^{-4}
CPPA	15 000	94.200	1.7928×10^{-3}	3.3230×10^{-4}

Numerical Example: Riemanniann Median on $\mathcal{S}^d$

- $\mathcal{S}^d$ $d-$dimensional sphere
- $\bar{p}$ north pole
- $q^{(1)}, \ldots, q^{(n)} \in \mathcal{S}^d$ are $n = 1000$ Gaussian random data points in $B_{\frac{\pi}{8}}(\bar{p})$
- $\mathcal{D} = \{q^{(1)}, \ldots, q^{(n)} \mid q^{(j)} \in \mathcal{S}^d \text{ for all } j = 1, \ldots, n\}$

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Solve

$$p^* := \operatorname*{arg\,min}_{p \in \mathcal{S}^d} f(p)$$

Riemanniann Median on $\mathcal{S}^d$ - Algorithms

	RCBM			PBA		
Dimension	Iter.	Time (sec.)	Objective	Iter.	Time (sec.)	Objective
2	19	6.50×10^{-3}	0.192 89	20	5.30×10^{-3}	0.192 89
4	28	1.01×10^{-2}	0.198 81	23	5.99×10^{-3}	0.198 81
32	58	2.29×10^{-2}	0.195 76	28	1.13×10^{-2}	0.195 76
1024	48	3.91×10^{-1}	0.197 75	40	3.31×10^{-1}	0.197 75
32 768	43	7.54	0.192 90	21	4.16	0.192 90

	SGM		
Dimension	Iter.	Time (sec.)	Objective
2	5000	1.14	0.192 89
4	3270	8.09×10^{-1}	0.198 81
32	5000	2.18	0.195 76
1024	122	9.75×10^{-1}	0.197 75
32 768	172	5.25×10^{1}	0.192 90

Conclusion and Future Work

In summary:

- introduced the Riemannian Convex Bundle Method for geodesically convex functions on Riemannian manifolds
- discussed convergence and related challenges
- showed two numerical examples

To do:

- further investigate the implications of positive curvature

Selected References

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