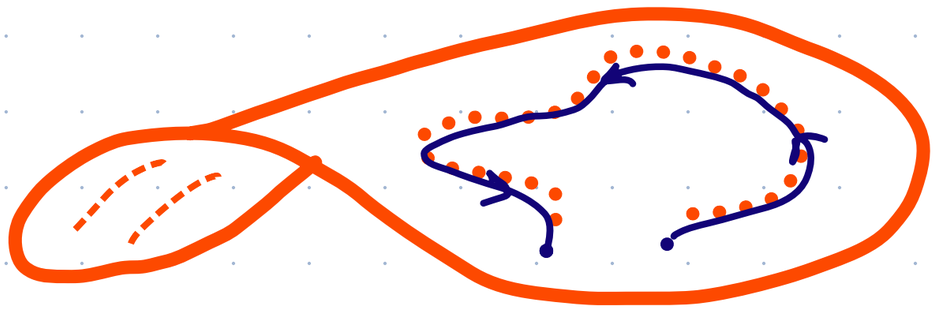
simula

Riemannian Metric Preconditioning for Trajectory Tracking



HAJG JASA (SIMULA)

joint work with JACOB GOODMAN (NTNU)

National Matematikermøte, Trondheim, 9-11 September 2026

simula

Motivation

- Trajectory tracking is "natural" on nonlinear spaces

simula

Motivation

- Trajectory tracking is "natural" on nonlinear spaces
  - ↳ DOFs of robots belong to e.g. rotoluanslations

simula

Motivation

- Trajectory tracking is "natural" on nonlinear spaces
  - ↳ DOFs of robots belong to e.g. rototranslations
- Different metrics g determine different kinetic energies

simula

Motivation

- Trajectory tracking is "natural" on nonlinear spaces
  ↳ DOFs of robots belong to e.g. rototranslations
- Different metrics g determine different kinetic energies
  ↳ this depends on the Levi-Civita connection ∇

simula

Motivation

- Trajectory tracking is "natural" on nonlinear spaces
  ↳ DOFs of robots belong to e.g. rototranslations
- Different metrics g determine different kinetic energies
  ↳ this depends on the Levi-Civita connection ∇
- Inspired by a metric modification strategy for obstacle avoidance

simula

Motivation

- Trajectory tracking is "natural" on nonlinear spaces
  - ↳ DOFs of robots belong to e.g. rotolranslations
- Different metrics g determine different kinetic energies
  - ↳ this depends on the Levi-Civita connection ∇
- Inspired by a metric modification strategy for obstacle avoidance
  - ↳ Gauge functions naturally deform the space

simula

Notation

. Riemannian manifold : $\mathcal{M}$

simula

Notation

- Riemannian manifold: M
- musical isomorphisms: $\cdot^{\flat}: TM \to T^*M$, $\cdot^{\#}: T^*M \to TM$

simula

Notation

- Riemannian manifold: M
- musical isomorphisms: $\cdot^{\flat}: TM \to T^*M$, $\cdot^{\#}: T^*M \to TM$
- vector fields: $X \in \mathfrak{X}(M)$; 1-forms: $\mu \in \Omega^1(M)$

simula

Notation

- Riemannian manifold: M
- musical isomorphisms: $\cdot^\flat : TM \to T^*M$, $\cdot^\sharp : T^*M \to TM$
- vector fields: $X \in \mathfrak{X}(M)$; 1-forms: $\mu \in \Omega^1(M)$
- cometric: $g^*(X^\flat, Y^\flat) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$

simula

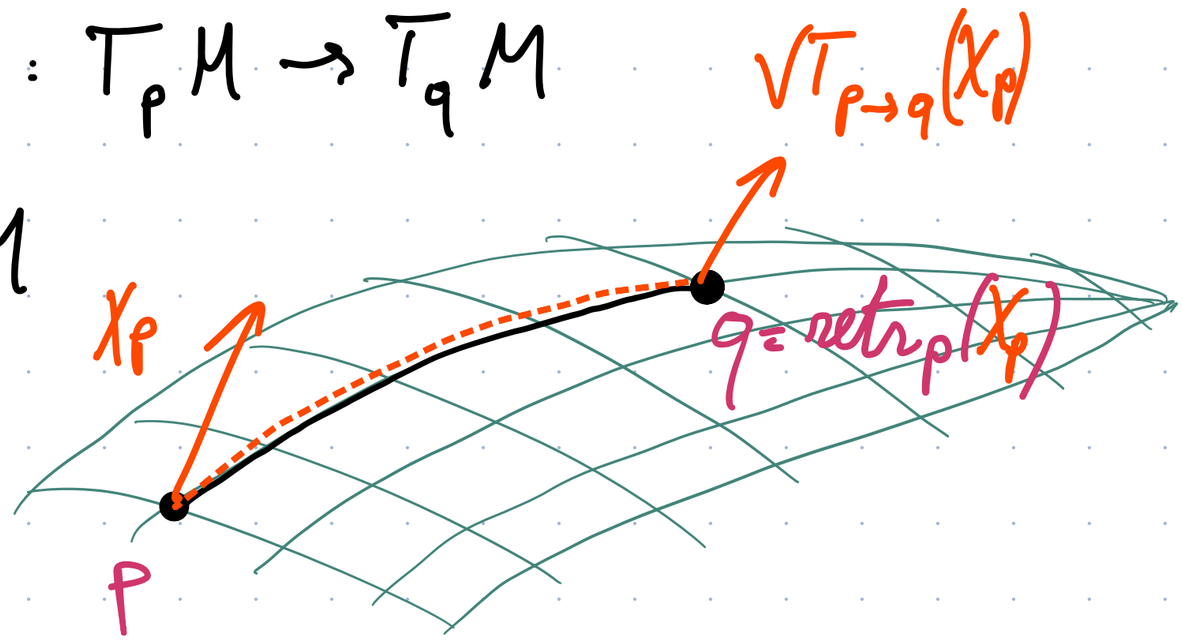
Notation

- Riemannian manifold: M
- musical isomorphisms: $\cdot^b : TM \to T^*M$, $\cdot^\# : T^*M \to TM$
- vector fields: $X \in \mathfrak{X}(M)$; 1-forms: $\mu \in \Omega^1(M)$
- cometric: $g^*(X^b, Y^b) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$
- vector transport: $VT_{p \to q} : T_pM \to T_qM$

simula

Notation

- Riemannian manifold: M
- musical isomorphisms: $\cdot^{\flat}: TM \to T^*M$, $\cdot^{\#}: T^*M \to TM$
- vector fields: $X \in \mathfrak{X}(M)$; 1-forms: $\mu \in \Omega^1(M)$
- cometric: $g^*(X^{\flat}, Y^{\flat}) = g(X, Y)$, $X, Y \in \mathfrak{X}(M)$
- vector transport: $VT_{p \to q} : T_pM \to T_qM$
- retraction: $retr_p : T_pM \to M$



simula

Reweighted metric(s)

Given a metric g, cometric g^*, and $\mu \in \Omega^1(M)$,

simula

Rewound metric(s)

Given a metric g, cometric g^*, and $\mu \in \Omega^1(M)$,

$$g^*_\mu := g^* + \mu^\# \otimes \mu^\# \quad \text{on} \quad T^*M$$

simula

Reward metric(s)

Given a metric g, cometric g^*, and $\mu \in \Omega^1(M)$,

$$g^*_\mu := g^* + \mu^\# \otimes \mu^\# \quad \text{on} \quad T^*M$$

induces

$$g_\mu = g - \frac{\mu \otimes \mu}{1 + \|\mu^\#\|_g^2} \quad \text{a reward metric on } TM$$

simula

Reward metric(s)

Given a metric g, cometric g^*, and $\mu \in \Omega^1(M)$,

$$g^*_\mu := g^* + \mu^\# \otimes \mu^\# \quad \text{on} \quad T^*M$$

induces

$$g_\mu = g - \frac{\mu \otimes \mu}{1 + \|\mu^\#\|^2_g} \quad \text{a reward metric on } TM$$

⚠ A modification $g + \mu \otimes \mu$, which induced g^*_μ, *inflates* distances in the direction of $\mu^\#$

simula

Reward metric(s)

Given a metric g, cometric g^*, and $\mu \in \Omega^1(M)$,

$$g^*_\mu := g^* + \mu^\# \otimes \mu^\# \quad \text{on} \quad T^*M$$

induces

$$g_\mu = g - \frac{\mu \otimes \mu}{1 + \|\mu^\#\|_g^2} \quad \text{a reward metric on } TM$$

⚠ A modification $g + \mu \otimes \mu$, which induced g^*_μ, *inflates* distances in the direction of $\mu^\#$

But

$$g_\mu(X, X) = \|X\|_g^2 - \frac{\mu(X)^2}{1 + \|\mu^\#\|_g^2}$$

simula

Reward metric(s)

Given a metric g, cometric g^*, and $\mu \in \Omega^1(M)$,

$$g^*_\mu := g^* + \mu^\# \otimes \mu^\# \quad \text{on} \quad T^*M$$

induces

$$g_\mu = g - \frac{\mu \otimes \mu}{1 + \|\mu^\#\|_g^2} \quad \text{a reward metric on } TM$$

⚠ A modification $g + \mu \otimes \mu$, which induced g^*_μ, *inflates* distances in the direction of $\mu^\#$

But

$$g_\mu(X,X) = \|X\|_g^2 - \frac{\mu(X)^2}{1+\|\mu^\#\|_g^2} \leq \|X\|_g^2$$

simula

Reward metric(s)

Given a metric g, cometric g^*, and $\mu \in \Omega^1(M)$,

$$g^*_\mu := g^* + \mu^\# \otimes \mu^\# \quad \text{on} \quad T^*M$$

induces

$$g_\mu = g - \frac{\mu \otimes \mu}{1 + \|\mu^\#\|_g^2} \quad \text{a reward metric on } TM$$

⚠ A modification $g + \mu \otimes \mu$, which induced g^*_μ,

inflates distances in the direction of $\mu^\#$

But

$$g_\mu(X,X) = \|X\|_g^2 - \frac{\mu(X)^2}{1+\|\mu^\#\|_g^2} \leq \|X\|_g^2$$

rewards motion in the direction of $\mu^\#$

simula

Completeness of $\mathcal{GM}$

Given a continuous, nondecreasing $V_0 : [0,\infty) \to (0,\infty)$ s.t.

simula

Completeness of g_μ

Given a continuous, nondecreasing $V_0 : [0,\infty) \to (0,\infty)$ s.t.

$$\int_0^\infty \frac{dx}{\sqrt{V_0(x)}} = +\infty ,$$

if g is a complete Riemannian metric on M, and $\mu \in \Omega^1(M)$ s.t.

simula

Completeness of g_μ

Given a continuous, nondecreasing $V_0 : [0,\infty) \to (0,\infty)$ s.t.

$$\int_0^\infty \frac{dx}{\sqrt{V_0(x)}} = +\infty ,$$

if g is a complete Riemannian metric on M, and $\mu \in \Omega^1(M)$ s.t.

$$1 + \|\mu^\#(p)\|_g^2 \le V_0(d_g(p,x_0)) \qquad \forall p \in M$$

for some $x_0 \in M$,

Completeness of g_μ

Given a continuous, nondecreasing $V_0 : [0,\infty) \to (0,\infty)$ s.t.

$$\int_0^\infty \frac{dx}{\sqrt{V_0(x)}} = +\infty ,$$

if g is a complete Riemannian metric on M, and $\mu \in \Omega^1(M)$ s.t.

$$1 + \|\mu^\#(p)\|_g^2 \leq V_0(d_g(p, x_0)) \quad \forall p \in M$$

for some $x_0 \in M$, then g_μ is complete.

Completeness of g_μ

Given a continuous, nondecreasing $V_0 : [0,\infty) \to (0,\infty)$ s.t.

$$\int_0^\infty \frac{dx}{\sqrt{V_0(x)}} = +\infty ,$$

if g is a complete Riemannian metric on M, and $\mu \in \Omega^1(M)$ s.t.

$$1 + \|\mu^\#(p)\|_g^2 \leq V_0(d_g(p, x_0)) \quad \forall p \in M$$

for some $x_0 \in M$, then g_μ is complete.

PROOF (sketch):

simula

Completeness of g_μ

Given a continuous, nondecreasing $V_0 : [0,\infty) \to (0,\infty)$ s.t.

$$\int_0^\infty \frac{dx}{\sqrt{V_0(x)}} = +\infty ,$$

if g is a complete Riemannian metric on M, and $\mu \in \Omega^1(M)$ s.t.

$$1 + \|\mu^\#(p)\|_g^2 \leq V_0(d_g(p, x_0)) \quad \forall p \in M$$

for some $x_0 \in M$, then g_μ is complete.

PROOF (sketch):

1) g_μ dominates the conformal metric $\frac{1}{1+\|\mu^\#\|_g^2} g$

simula

Completeness of g_μ

Given a continuous, nondecreasing $V_0 : [0,\infty) \to (0,\infty)$ s.t.

$$\int_0^\infty \frac{dx}{\sqrt{V_0(x)}} = +\infty \,,$$

if g is a complete Riemannian metric on M, and $\mu \in \Omega^1(M)$ s.t.

$$1 + \|\mu^\#(p)\|_g^2 \leq V_0(d_g(p, x_0)) \quad \forall p \in M$$

for some $x_0 \in M$, then g_μ is complete.

PROOF (sketch):

1) g_μ dominates the conformal metric $\frac{1}{1+\|\mu^\#\|_g^2} g =: fg$

2) fg is complete by contradiction, using the growth condition

simula

Levi-Civita Connection

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the LC-CC of g, g_μ resp.

simula

Levi-Civita Connection

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the LC-CC of g, g_μ resp. For any $X, Y \in \mathfrak{X}(M)$ and

$$\lambda := 1 + \|\mu^\sharp\|_g^2 \,, \quad \mu_\perp^\sharp := \frac{\mu(X)\mu(Y)}{\lambda}\mu^\sharp - \frac{\mu(Y)}{2}X - \frac{\mu(X)}{2}Y$$

Levi-Civita Connection

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the LC-CC of g, g_μ resp. For any $X, Y \in \mathfrak{X}(M)$ and

$$\lambda := 1 + \|\mu^\sharp\|_g^2 \,, \quad \mu_\perp^\sharp := \frac{\mu(X)\mu(Y)}{\lambda}\mu^\sharp - \frac{\mu(Y)}{2}X - \frac{\mu(X)}{2}Y$$

$$\tilde{\nabla}_X Y = \nabla_X Y$$

Levi-Civita Connection

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the LC-CC of g, g_μ resp. For any $X, Y \in \mathfrak{X}(M)$ and

$$\lambda := 1 + \|\mu^\#\|_g^2 \,, \quad \mu_\perp^\# := \frac{\mu(X)\mu(Y)}{\lambda}\mu^\# - \frac{\mu(Y)}{2}X - \frac{\mu(X)}{2}Y$$

$$\tilde{\nabla}_X Y = \nabla_X Y + \left[\frac{1}{\lambda}\langle \mu(Y)\nabla_X\mu^\# + \mu(X)\nabla_Y\mu^\#, \mu^\#\rangle - \langle \nabla_X\mu^\#, Y\rangle\right]\mu^\#$$

simula

Levi-Civita Connection

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the Levi-Civita connections of g, g_μ respectively. Then for any $X, Y \in \mathfrak{X}(M)$ and

$$\lambda := 1 + \|\mu^\sharp\|_g^2 \quad , \quad \mu_\perp^\sharp := \frac{\mu(X)\mu(Y)}{\lambda}\mu^\sharp - \frac{\mu(Y)}{2}X - \frac{\mu(X)}{2}Y$$

$$\tilde{\nabla}_X Y = \nabla_X Y + \left[\frac{1}{\lambda}\left\langle \mu(Y)\nabla_X \mu^\sharp + \mu(X)\nabla_Y \mu^\sharp, \mu^\sharp \right\rangle - \left\langle \nabla_X \mu^\sharp, Y \right\rangle\right]\mu^\sharp$$

$$+ \left[\frac{1}{2} d\mu(X,Y) - \frac{\mu(X)\mu(Y)}{\lambda}\mu\left(\frac{\nabla_{\mu^\sharp}\mu^\sharp}{\|\mu^\sharp\|^2}\right) + \frac{\lambda\, d\mu(\mu_\perp^\sharp, \mu^\sharp)}{\|\mu^\sharp\|^2}\right]\mu^\sharp$$

Levi-Civita Connection

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the LC-CC of g, g_μ resp. For any $X, Y \in \mathfrak{X}(M)$ and

$$\lambda := 1 + \|\mu^\sharp\|_g^2 \,, \quad \mu_\perp^\sharp := \frac{\mu(X)\mu(Y)}{\lambda}\mu^\sharp - \frac{\mu(Y)}{2}X - \frac{\mu(X)}{2}Y$$

$$\tilde{\nabla}_X Y = \nabla_X Y + \left[\frac{1}{\lambda}\langle \mu(Y)\nabla_X \mu^\sharp + \mu(X)\nabla_Y \mu^\sharp, \mu^\sharp\rangle - \langle \nabla_X \mu^\sharp, Y\rangle\right]\mu^\sharp$$

$$+\left[\frac{1}{2}d\mu(X,Y) - \frac{\mu(X)\mu(Y)}{\lambda}\mu\left(\frac{\nabla_{\mu^\sharp}\mu^\sharp}{\|\mu^\sharp\|^2}\right) + \frac{\lambda\, d\mu(\mu_\perp^\sharp, \mu^\sharp)}{\|\mu^\sharp\|^2}\right]\mu^\sharp$$

$$-\frac{\mu(X)\mu(Y)}{\lambda^2}\left[\nabla_{\mu^\sharp}\mu^\sharp - \frac{\mu\left(\nabla_{\mu^\sharp}\mu^\sharp\right)}{\|\mu^\sharp\|^2}\mu^\sharp\right]$$

simula

Levi-Civita Connection

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the Levi-Civita connections of g, g_μ respectively. For any $X, Y \in \mathfrak{X}(M)$ and

$$\lambda := 1 + \|\mu^\sharp\|_g^2 \,, \qquad \mu_\perp^\sharp := \frac{\mu(X)\mu(Y)}{\lambda}\mu^\sharp - \frac{\mu(Y)}{2}X - \frac{\mu(X)}{2}Y$$

$$\begin{aligned}
\tilde{\nabla}_X Y = \nabla_X Y &+ \left[\frac{1}{\lambda}\left\langle \mu(Y)\nabla_X \mu^\sharp + \mu(X)\nabla_Y \mu^\sharp, \mu^\sharp \right\rangle - \left\langle \nabla_X \mu^\sharp, Y \right\rangle\right]\mu^\sharp \\
&+ \left[\frac{1}{2} d\mu(X,Y) - \frac{\mu(X)\mu(Y)}{\lambda}\mu\left(\frac{\nabla_{\mu^\sharp}\mu^\sharp}{\|\mu^\sharp\|^2}\right) + \frac{\lambda\, d\mu(\mu_\perp^\sharp, \mu^\sharp)}{\|\mu^\sharp\|^2}\right]\mu^\sharp \\
&- \frac{\mu(X)\mu(Y)}{\lambda^2}\left[\nabla_{\mu^\sharp}\mu^\sharp - \frac{\mu\left(\nabla_{\mu^\sharp}\mu^\sharp\right)}{\|\mu^\sharp\|^2}\mu^\sharp\right] \\
&+ \frac{1}{\lambda} d\mu(\mu_\perp^\sharp, \cdot)^\sharp - \frac{d\mu(\mu_\perp^\sharp, \mu^\sharp)}{\|\mu^\sharp\|^2}\mu^\sharp
\end{aligned}$$

simula

Levi-Civita Connection (it's complicated...)

Let $\mu \in \Omega^1(M) \setminus \{0\}$ and let $\nabla, \tilde{\nabla}$ be the LC-CC of g, g_μ resp. For any $X, Y \in \mathfrak{X}(M)$ and

$$\lambda := 1 + \|\mu^\sharp\|_g^2 \,, \quad \mu_\perp^\sharp := \frac{\mu(X)\mu(Y)}{\lambda}\mu^\sharp - \frac{\mu(Y)}{2}X - \frac{\mu(X)}{2}Y$$

$$\tilde{\nabla}_X Y = \nabla_X Y + \Big[\frac{1}{\lambda}\langle \mu(Y)\nabla_X\mu^\sharp + \mu(X)\nabla_Y\mu^\sharp, \mu^\sharp\rangle - \langle \nabla_X\mu^\sharp, Y\rangle\Big]\mu^\sharp$$
$$+\Big[\frac{1}{2}d\mu(X,Y) - \frac{\mu(X)\mu(Y)}{\lambda}\frac{\mu(\nabla_{\mu^\sharp}\mu^\sharp)}{\|\mu^\sharp\|^2} + \frac{\lambda\, d\mu(\mu_\perp^\sharp, \mu^\sharp)}{\|\mu^\sharp\|^2}\Big]\mu^\sharp$$
$$-\frac{\mu(X)\mu(Y)}{\lambda^2}\Big[\nabla_{\mu^\sharp}\mu^\sharp - \frac{\mu(\nabla_{\mu^\sharp}\mu^\sharp)}{\|\mu^\sharp\|^2}\mu^\sharp\Big]$$
$$+\frac{1}{\lambda}d\mu(\mu_\perp^\sharp, \cdot)^\sharp - \frac{d\mu(\mu_\perp^\sharp, \mu^\sharp)}{\|\mu^\sharp\|^2}\mu^\sharp$$

stuff

$T_X Y$

Affine Connection Control Systems

Given (M, g) with $\mathcal{L}$-CC ∇, an ACCS with control distribution $\mathcal{D} \subset TM$ takes the form

$$\nabla_{\dot{q}} \dot{q} = Y_0(q(t)) + \sum_{j=1}^{k} u^j(t) Y_j(q(t))$$

simula

Affine Connection Control Systems

Given (M, g) with $\mathcal{L}$-CC ∇, an ACCS with control distribution $\mathcal{D} \subset TM$ takes the form

$$\nabla_{\dot{q}} \dot{q} = Y_0(q(t)) + \sum_{j=1}^{k} u^j(t) Y_j(q(t))$$

with a drift $Y_0 \in \mathfrak{X}(M)$, $Y_j \in \mathfrak{X}(M)$ span $\mathcal{D}$, and $u^j \in C^0([0, t_f])$ are the control inputs for some final time $t_f > 0$

simula

Affine Connection Control Systems

Given (M, g) with $\mathcal{L}$-CC ∇, an ACCS with control distribution $\mathcal{D} \subset TM$ takes the form

$$\nabla_{\dot{q}} \dot{q} = Y_0(q(t)) + \sum_{j=1}^{k} u^j(t) Y_j(q(t))$$

with a drift $Y_0 \in \mathfrak{X}(M)$, $Y_j \in \mathfrak{X}(M)$ span $\mathcal{D}$, and $u^j \in C^0([0, t_f])$ are the control inputs for some final time $t_f > 0$

If $q_d \in C^2([0, t_f])$ is a desired trajectory, the task is to design the $u^j(t)$ s.t. the integral curves of ACCS track q_d on $[0, t_f]$

Affine Connection Control Systems (continued)

To assess tracking quality, configuration errors are defined:

$$e_q := -\mathrm{retr}_q^{-1} q_d \in T_q M$$

Affine Connection Control Systems (continued)

To assess tracking quality, configuration errors are defined:

$$e_q := -\mathrm{retr}_q^{-1} q_d \in T_q M$$

$$e_v := \dot{q} - VT_{q_d \to q} \dot{q}_d \in T_q M$$

Affine Connection Control Systems (continued)

To assess tracking quality, configuration errors are defined:

$$e_q := -\mathrm{retr}_q^{-1} q_d \in T_q M$$

$$e_v := \dot{q} - VT_{q_d \to q} \dot{q}_d \in T_q M$$

satisfying

$$e_q = e_v = 0 \iff q = q_d \quad \text{(locally)}$$

simula

Affine Connection Control Systems (continued)

To assess tracking quality, configuration errors are defined:

$$e_q := -\mathrm{retr}_q^{-1} q_d \in T_q M$$

$$e_v := \dot{q} - VT_{q_d \to q} \dot{q}_d \in T_q M$$

satisfying $e_q = e_v = 0 \iff q = q_d$ (locally)

→ Use a proportional derivative (PD) control

$$u := VT_{q_d \to q}\left(D_{\dot{q}_d} \dot{q}_d\right) - k_q e_q - k_v e_v$$

Affine Connection Control Systems (continued)

To assess tracking quality, configuration errors are defined:

$$e_q := -\mathrm{retr}_q^{-1} q_d \in T_q M$$

$$e_v := \dot{q} - VT_{q_d \to q} \dot{q}_d \in T_q M$$

satisfying $e_q = e_v = 0 \iff q = q_d$ (locally)

→ Use a proportional derivative (PD) control

$$u := VT_{q_d \to q}\left(\nabla_{\dot{q}_d} \dot{q}_d\right) - k_q e_q - k_v e_v$$

with gains $k_q, k_v \geq 0$

simula

Geometric Preconditioner

Choose $X \in \mathcal{X}(M)$, $\lambda \geq 0$ and $\mu := \lambda X^b$

simula

Geometric Preconditioner

Choose $X \in \mathcal{X}(M)$, $\lambda \geq 0$ and $\mu := \lambda X^\flat$, then

$$\|Y\|_{g_\mu} = \|Y\| \quad \text{if} \quad Y \perp_g X$$

Geometric Preconditioner

Choose $X \in \mathfrak{X}(\Omega)$, $\lambda \geq 0$ and $\mu := \lambda X^b$, then

$$\|Y\|_{g_\mu} = \|Y\| \quad \text{if} \quad Y \perp_g X,$$

but

$$\|X\|_{g_\mu}^2 \to 0 \quad \text{as} \quad \lambda \to \infty$$

Geometric Precondition

Choose $X \in \mathfrak{X}(M)$, $\lambda \geq 0$ and $\mu := \lambda X^\flat$, then

$$\|Y\|_{g_\mu} = \|Y\| \quad \text{if} \quad Y \perp_g X \,, \text{ but}$$

$$\|X\|^2_{g_\mu} \to 0 \quad \text{as} \quad \lambda \to \infty$$

Writing $\hat{\nabla}^\lambda_{\dot{q}} \dot{q} = \nabla_{\dot{q}} \dot{q} + T_{\dot{q}} \dot{q}$ in ACCS

simula

Geometric Precondition

Choose $X \in \mathfrak{X}(\mathcal{M})$, $\lambda \geq 0$ and $\mu := \lambda X^\flat$, then

$$\|Y\|_{g_\mu} = \|Y\| \quad \text{if} \quad Y \perp_g X \,, \text{ but}$$

$$\|X\|^2_{g_\mu} \to 0 \quad \text{as} \quad \lambda \to \infty$$

Writing $\hat{\nabla}^\lambda_{\dot{q}}\dot{q} = \nabla_{\dot{q}}\dot{q} + T_{\dot{q}}\dot{q}$ in ACCS,

$$\Rightarrow \quad u_{pc} := u + VT_{q_d \to q}\left(T^\lambda_{\dot{q}_d}\dot{q}^d\right) - T^\lambda_{\dot{q}}\dot{q}$$

simula

Geometric Precondition

Choose $X \in \mathfrak{X}(M)$, $\lambda \geq 0$ and $\mu := \lambda X^\flat$, then

$$\|Y\|_{g_\mu} = \|Y\| \quad \text{if} \quad Y \perp_g X \,, \quad \text{but}$$

$$\|X\|^2_{g_\mu} \to 0 \quad \text{as} \quad \lambda \to \infty$$

Writing $\hat{\nabla}^\lambda_{\dot{q}}\dot{q} = \nabla_{\dot{q}}\dot{q} + T_{\dot{q}}\dot{q}$ in ACCS,

$$\Rightarrow \quad u_{pc} := u + VT_{q_d \to q}\left(T^\lambda_{\dot{q}_d}\dot{q}^d\right) - T^\lambda_{\dot{q}}\dot{q}$$

and if $D \neq TM$, $\quad u^D_{pc} := \Pi_D(u_{pc})$

Numerical Experiment

Setup: $SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, R \in SO(3),\ p \in \mathbb{R}^3 \right\}$

Numerical Experiment

Setup: $SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, R \in SO(3),\ p \in \mathbb{R}^3 \right\}$

$$\mathfrak{se}(3) = \left\{ \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \hat{\Omega} \in \mathfrak{so}(3),\ v \in \mathbb{R}^3 \right\}$$

simula

Numerical Experiment

Setup: $SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} \,\middle|\, R \in SO(3),\ p \in \mathbb{R}^3 \right\}$

$$se(3) = \left\{ \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \,\middle|\, \hat{\Omega} \in so(3),\ v \in \mathbb{R}^3 \right\}$$

$$\dot{R} = R\hat{\Omega} \qquad \dot{p} = Rv$$

$$\dot{v} = -\Omega \times v + f e_3 \qquad \dot{\Omega} = \mathbb{J}^{-1}(\mathbb{J}\Omega \times \Omega) + \tau$$

} rigid body

simula

Numerical Experiment

Setup: $SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} \,\middle|\, R \in SO(3),\ p \in \mathbb{R}^3 \right\}$

$$se(3) = \left\{ \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \,\middle|\, \hat{\Omega} \in so(3),\ v \in \mathbb{R}^3 \right\}$$

$$\dot{R} = R\hat{\Omega} \qquad \dot{p} = Rv$$

$$\dot{v} = -\Omega \times v + f e_3 \qquad \dot{\Omega} = \mathbb{J}^{-1}(\mathbb{J}\Omega \times \Omega) + \tau$$

with controls $f \in \mathbb{R}$, $\tau \in \mathbb{R}^3$

simula

Numerical Experiment

Setup: $SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4} \,\middle|\, R \in SO(3),\ p \in \mathbb{R}^3 \right\}$

$$se(3) = \left\{ \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \,\middle|\, \hat{\Omega} \in so(3),\ v \in \mathbb{R}^3 \right\}$$

$$\dot{R} = R\hat{\Omega} \qquad \dot{p} = Rv$$

$$\dot{v} = -\Omega \times v + f e_3 \qquad \dot{\Omega} = \mathbb{J}^{-1}(\mathbb{J}\Omega \times \Omega) + \tau$$

with controls $f \in \mathbb{R}$, $\tau \in \mathbb{R}^3$

→ that's where the preconditioning comes in

simula

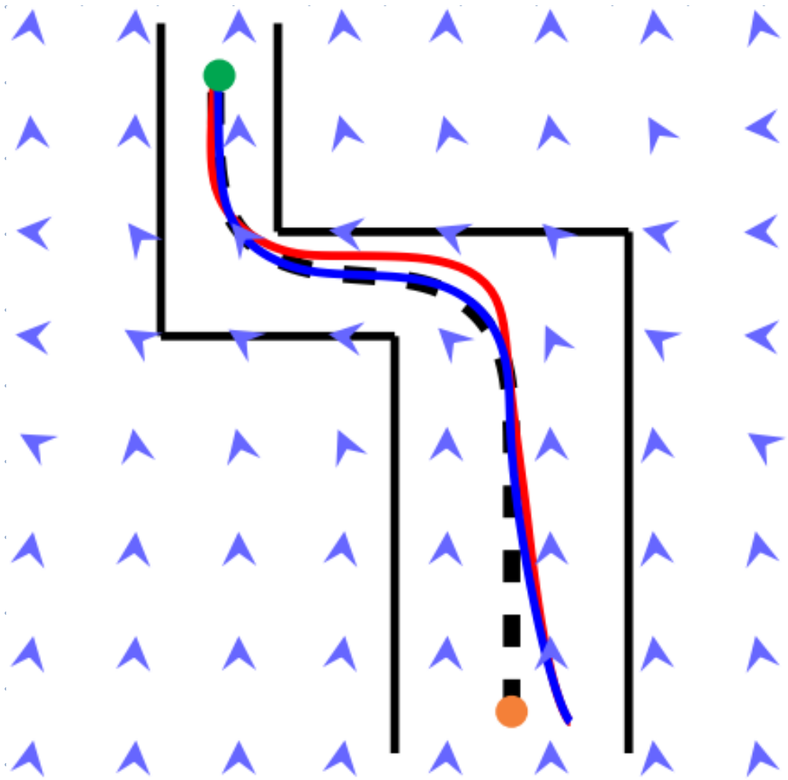
Numerical Experiment (continued...)

Design a vector field X that navigates the maze whose integral curve (1) is the desired trajectory q_d and use $\mu = X^\flat$ for g_r.

simula

Numerical Experiment (continued...)

Design a vector field X that navigates the maze whose integral curve (1) is the desired trajectory q_d and use $\mu = X^\flat$ for g_μ.

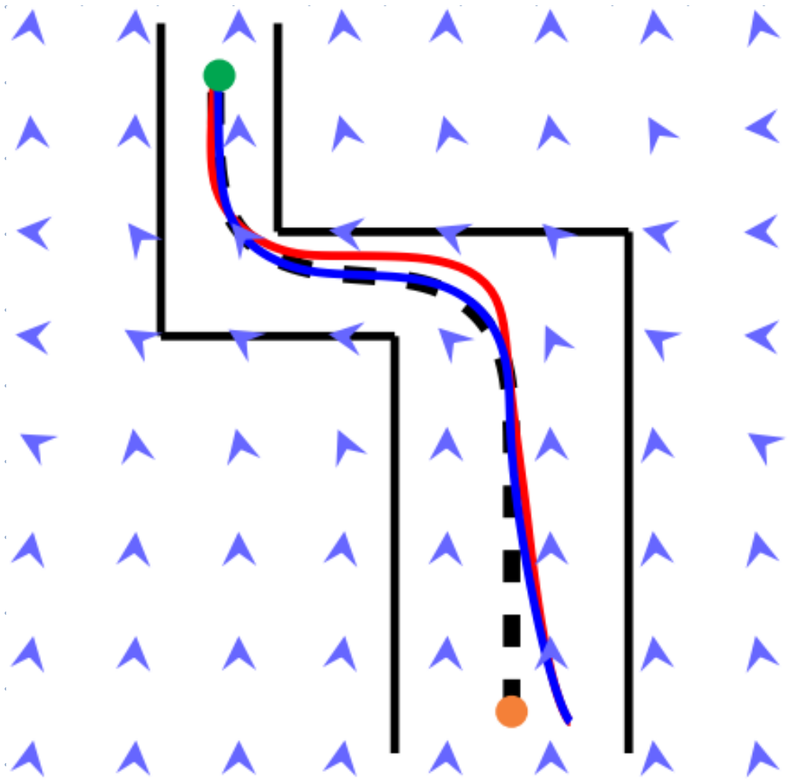


(1) Preconditioned, Euclidean

simula

Numerical Experiment (continued...)

Design a vector field X that navigates the maze whose integral curve (1) is the desired trajectory q_d and use $\mu = X^\flat$ for g_μ. Then test also with a q_d that is not integral (2) for X.

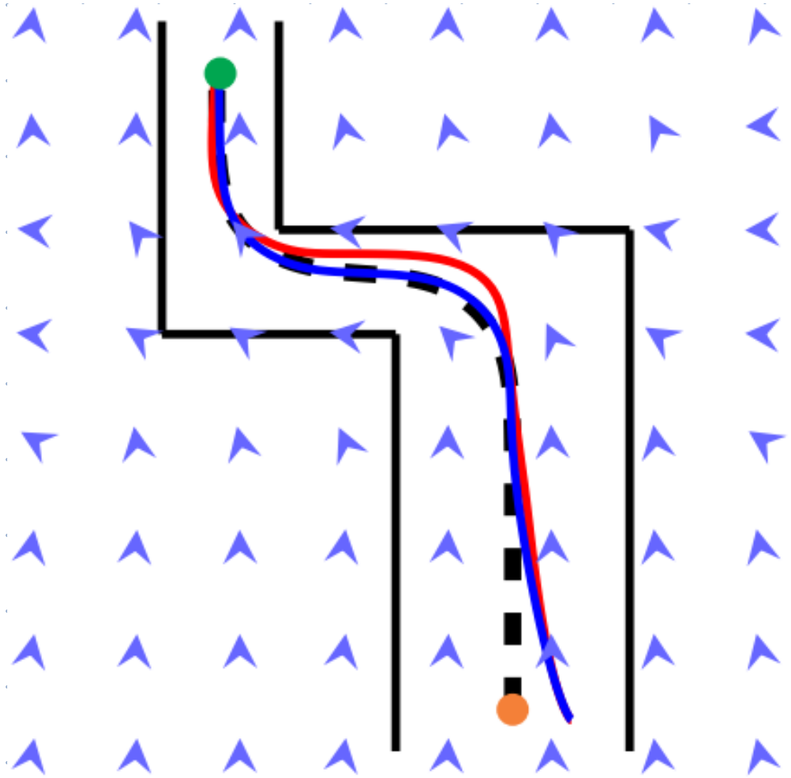


(1) Preconditioned, Euclidean

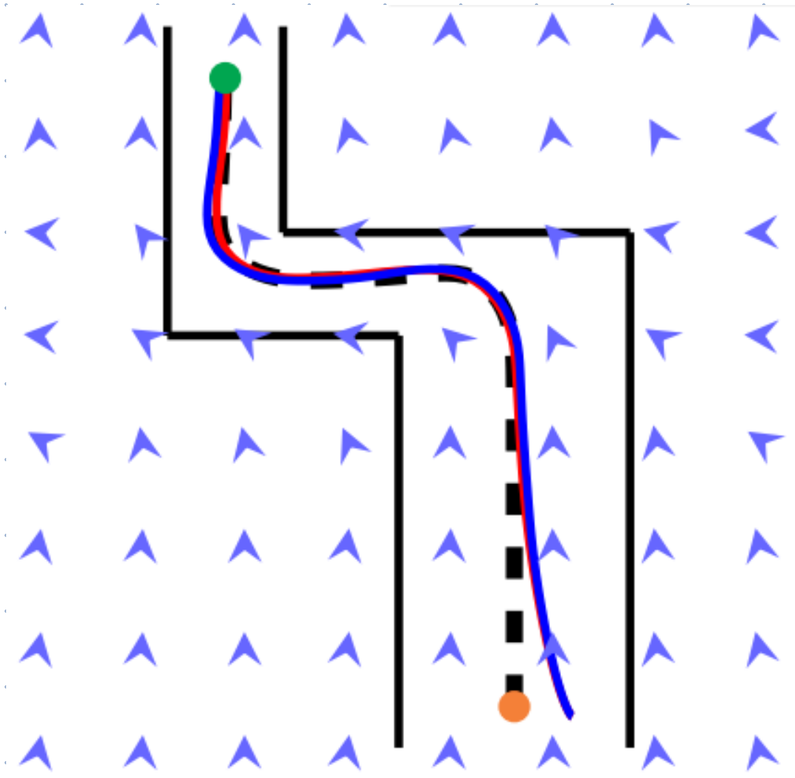
simula

Numerical Experiment (continued...)

Design a vector field X that navigates the maze whose integral curve (1) is the desired trajectory q_d and use $\mu = X^\flat$ for g_μ. Then test also with a q_d that is not integral (2) for X.



(1) Preconditioned, Euclidean



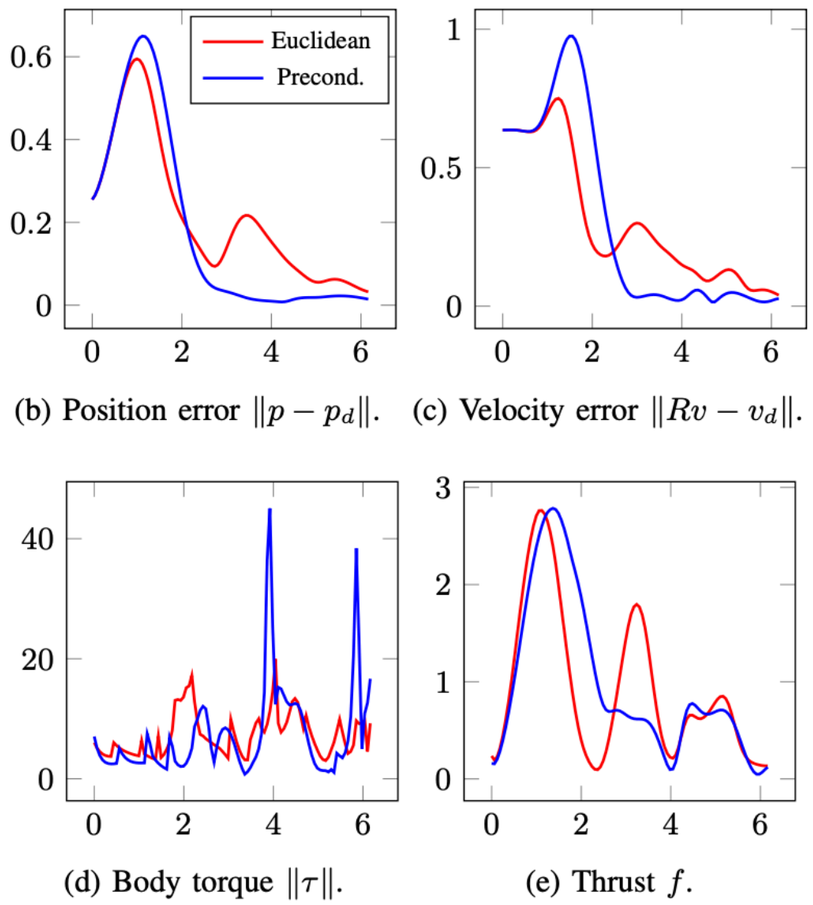
(2) Preconditioned, Euclidean

simula

Numerical Experiment (continued...)

Design a vector field X that navigates the maze whose integral curve (1) is the desired trajectory q_d and use $\mu = X^\flat$ for g_μ. Then test also with a q_d that is not integral (2) for X.

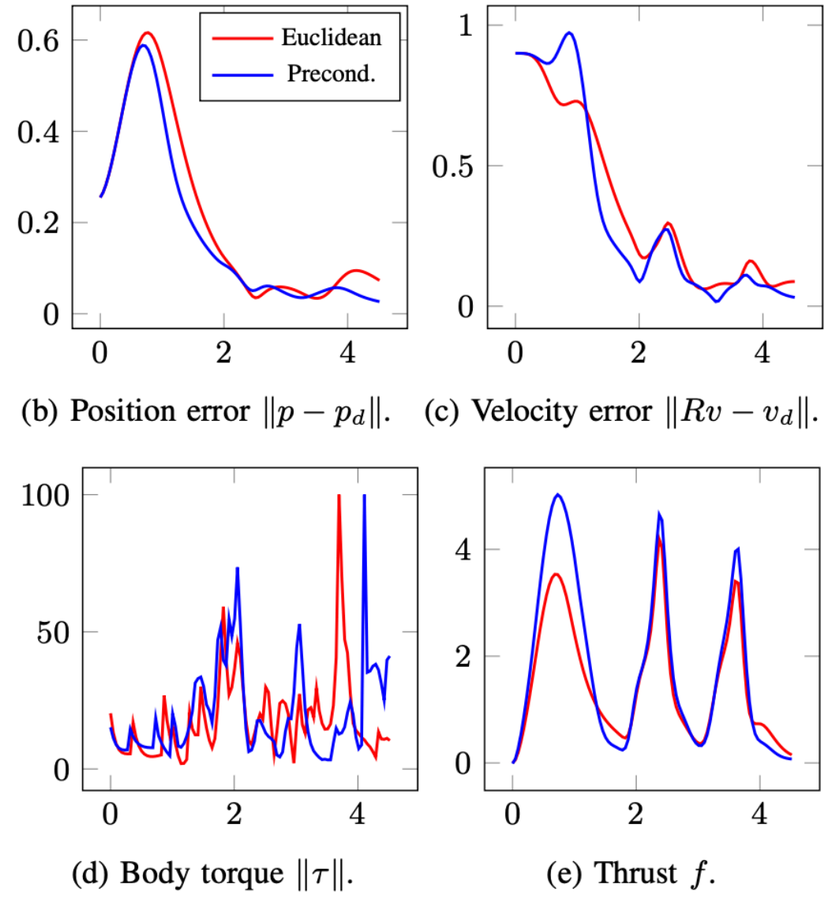
(1)



(b) Position error $\|p - p_d\|$. (c) Velocity error $\|Rv - v_d\|$.

(d) Body torque $\|\tau\|$. (e) Thrust f.

(2)



(b) Position error $\|p - p_d\|$. (c) Velocity error $\|Rv - v_d\|$.

(d) Body torque $\|\tau\|$. (e) Thrust f.

Thank you for your attention!

arXiv